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# Importance sampling for McKean-Vlasov SDEs

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This paper deals with the Monte-Carlo methods for evaluating expectations of functionals of solutions to McKean-Vlasov Stochastic Differential Equations (MV-SDE) with drifts of super-linear growth. We assume the MV-SDE is approximated by an interacting particle system and propose two importance sampling (IS) techniques to reduce the variance of the resulting Monte Carlo estimator. In the *complete measure change* approach, the IS measure change is applied simultaneously in the coefficients and in the expectation to be evaluated. In the *decoupling* approach we first estimate the law of the solution in a first set of simulations without measure change and then perform a second set of simulations under the importance sampling measure using the approximate solution law computed in the first step.

For both approaches, under a constant diffusion coefficient, we use large deviations techniques to identify an optimisation problem for the candidate measure change. The decoupling approach yields a far simpler optimisation problem than the complete measure change, however, we can reduce the complexity of the complete measure change through some symmetry arguments. We implement both algorithms for two examples coming from the Kuramoto model from statistical physics and show that the variance of the importance sampling schemes is up to 3 orders of magnitude smaller than that of the standard Monte Carlo. The computational cost is approximately the same as for standard Monte Carlo for the complete measure change and only increases by a factor of 2–3 for the decoupled approach. We also estimate the propagation of chaos error and find that this is dominated by the statistical error by one order of magnitude.

*Keywords:* McKean-Vlasov Stochastic Differential Equation, interacting particle system, Monte Carlo simulation, Importance sampling, Large deviations.

*MSC2010:* 65C05 (Monte Carlo methods), 65C30 (Stochastic differential and integral equations), 65C35 (Stochastic particle methods)

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# 1. Introduction

The aim of this paper is to develop efficient importance sampling algorithms for computing the expectations of functionals of solutions to McKean-Vlasov stochastic differential equations (MV-SDE). MV-SDEs are stochastic differential equations where the coefficients depend on the law of the solution, typically written in the following form:

$$\mathrm{d}X_t = b(t, X_t, \mu_t)\mathrm{d}t + \sigma(t, X_t, \mu_t)\mathrm{d}W_t, \quad X_0 = x_0,$$

where $\mu_t$ denotes the law of the process $X$ at time $t$, and $W$ is a standard Brownian motion. MV-SDEs, also known as mean-field equations, were originally introduced in physics to describe the movement of an individual particle amongst a large number of indistinguishable particles interacting through their mean field. They are now used in a variety of other domains, such as finance, economics, biology, population dynamics etc.

Development of algorithms for the simulation of MV-SDEs is a very active area of research. One of the earliest works to consider the error and computational complexity involved in simulating a MV-SDE was [5]. More recently [24], [26] and [11] among others (see references therein) developed more efficient methods for simulating MV-SDEs under Lipschitz coefficients or even stronger assumptions.

A common technique for the simulation of MV-SDEs is to use the interacting particle representation. Namely, we consider $N$ particles indexed by $i = 1, \dots, N$, where each $X^{i,N}$ satisfies the following SDE,

$$\mathrm{d}X_t^{i,N} = b\Big(t, X_t^{i,N}, \mu_t^N\Big)\mathrm{d}t + \sigma\Big(t, X_t^{i,N}, \mu_t^N\Big)\mathrm{d}W_t^i, \quad \mu_t^N(\mathrm{d}x) := \frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^{j,N}}(\mathrm{d}x), \quad (1.1)$$

where $X_0^{i,N} = x_0$, $\delta_{X_t^{j,N}}$ is the Dirac measure at point $X_t^{j,N}$, and the Brownian motions $W^i, i = 1, \dots, N$ are independent. The so-called propagation of chaos result (see, e.g., [8]) states that under sufficient conditions, as $N \to \infty$, for every $i$, the process $X^{i,N}$ converges to $X^i$, the solution of the MV-SDE driven by the Brownian motion $W^i$.

The system (1.1) is a system of ordinary SDE and can be discretized with one of the many available methods such as the Euler scheme. Let $X_t^{i,N,n}$ be the $i$-th component of the solution of (1.1), discretized on $[0, T]$ over $n$ steps. The quantity of interest, which, in our case is $\theta = \mathbb{E}[G(X)]$, will then be approximated by the Monte Carlo estimator

$$\hat{\theta}^{N,n} = \frac{1}{N}\sum_{i=1}^{N} G(X^{i,N,n}).$$

The precision of this approximation is affected by three sources of error.

- The statistical error, that is, the difference between $\hat{\theta}^{N,n}$ and $\mathbb{E}[G(X^{i,N,n})]$.
- The discretization error, that is, the difference between $\mathbb{E}[G(X^{i,N,n})]$ and $\mathbb{E}[G(X^{i,N})]$.
- The propagation of chaos error of approximating the MV-SDE with the interacting particle system, that is, the difference between $\mathbb{E}[G(X^{i,N})]$ and $\mathbb{E}[G(X)]$.

The discretization error of ordinary SDEs has been analyzed by many authors, and it is well known that, e.g., under the Lipschitz assumptions (plus extra regularity or ellipticity) the Euler scheme has weak convergence error of order $\frac{1}{n}$. It is of course also known that the standard deviation of the statistical error is of order of $1/\sqrt{N}$.

There has also been some work detailing the error from the propagation of chaos as a function of $N$. Essentially for $G$ and $X$ nice enough this error is also of the order $1/\sqrt{N}$, see for example [27], [4] or [2] for further details. In spite of this relatively slow convergence, many MV-SDEs have a reasonably "nice" dependence on the law, which makes the particle approximation a good technique. On the other hand, one often wants to consider *rare events* in the context of the MV-SDE, and in this realm the statistical error will dominate the propagation of chaos error. The focus of this paper is therefore on the statistical error of the Monte Carlo method. In view of the poor convergence of the standard Monte Carlo, it is typical to enhance the standard approach with a so-called *variance reduction* technique. Importance sampling, which is the focus of this paper, is one such technique. We will discuss the point of statistical against propagation of chaos error in more detail in Section 5.

Importance sampling is based on the following identity, valid for any probability measure $\mathbb{Q}$ (absolutely continuous with respect to $\mathbb{P}$).

$$\mathbb{E}[G(X)] = \mathbb{E}_{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}G(X)\right].$$

The variance of the Monte Carlo estimator obtained by simulating $X$ under the measure $\mathbb{Q}$ and correcting by the corresponding Radon-Nikodym density is different from that of the standard estimator, and can be made much smaller by a judicious choice of the sampling measure $\mathbb{Q}$.

Importance sampling is most effective in the context of *rare event simulation*, that is, when the probability $\mathbb{P}[G(X) > 0]$ is small. Since the theory of large deviations is concerned with the study of probabilities of rare events, it is natural to use measure changes appearing in or inspired by the large deviations theory for importance sampling. We refer, e.g., to [17] and references therein for a review of this approach and to [21], [25], [31] for specific applications to financial models. The large deviations theory, on the one hand, simplifies the computation of the candidate importance sampling measure, and on the other hand, allows to define its optimality in a rigorous asymptotic framework.

In this paper we develop importance sampling techniques for MV-SDE with constant diffusion coefficient. Our main contribution is two-fold. Firstly, we show how one can apply a change of measure to MV-SDEs, and propose two algorithms, which can carry this out: the *complete measure change* algorithm and the *decoupling* algorithm. In the complete measure change approach, the IS measure change is applied simultaneously in the coefficients and in the expectation to be evaluated. In the decoupling approach we first estimate the law of the solution in a first set of simulations without measure change and then perform a second set of simulations under the importance sampling measure using the approximate solution law computed in the first step.

Secondly, we use large deviations techniques to obtain an optimisation problem for the candidate measure change for both approaches. We focus on the class of Cameron-Martin transforms, under which the measure change is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \mathcal{E}\Big( \int_0^T f_t \mathrm{d}W_t \Big) := \exp\left( \int_0^T f_t \mathrm{d}W_t - \frac{1}{2}\int_0^T f_t^2 \mathrm{d}t \right), \tag{1.2}$$

where $f_t$ is a deterministic function. Following earlier works on the subject, we use the large deviations theory to construct a tractable proxy for the variance of $G(X)$ under the new measure. Of course, the presence of the interacting particle approximation introduces additional complexity at this point. Moreover, unlike the work of [25] which considered a very restrictive class of SDEs (the geometric Brownian motion), here we deal with a general class of MV-SDE where the drifts are of super-linear growth and satisfy a monotonicity type condition. This is important in practice since many MV-SDEs fall into this category, see the examples given in [2], [10]; our algorithms may be of use for the simulation from the invariant distribution with a finite-time approximation [3] or in simulating relevant quantities for mean-field games [9].

We then minimise the large deviations proxy to obtain a candidate optimal measure change for the two approaches that we consider. We find that the decoupling approach yields an easier optimisation problem than the complete measure change, which results in a high dimensional problem. However, by using exchangeability arguments the latter problem may be transformed into a far simpler two dimensional one. We implement both algorithms for two examples coming from the Kuramoto model from statistical physics and show that the variance of the importance sampling schemes is up to 3 orders of magnitude smaller than that of the standard Monte Carlo. Moreover, the computational cost only increases by a factor of 2 for the decoupling approach and is approximately the same as standard Monte Carlo for the complete measure change. We also estimate the propagation of chaos error and find that this is dominated by the statistical error by one order of magnitude after variance reduction. That being said, although the complete measure change appears to operate well in certain situations, it does rely on a change of measure which is not too "large". We come back to this point throughout.

Concerning the measure change paradigm, in this work we focus on deterministic (open loop) measure changes as opposed to stochastic (feedback) measure changes. This is a choice one faces when using importance sampling and there are advantages and disadvantages to both alternatives. As pointed out in [22], deterministic measure changes may lead to poor results in terms of variance reduction, however, the increase in computational time of the IS algorithm compared to standard MC is overall negligible. Stochastic measure changes as discussed in [17] give improved variance reduction in far greater generality, however, calculating the measure change is computationally burdensome, so the overall computational gain is less clear. As this is the first paper to marry importance sampling with MV-SDEs we start by analyzing deterministic measure changes and leave stochastic measure changes to future research. We provide precise conditions under which our deterministic measure change leads to an asymptotically optimal importance sampling estimator in the class of measure changes stemming from Cameron-Martin densities (deterministic). Further, the complete measure change algorithm requires a propagation of chaos result to hold *under* the new measure (Proposition 3.1) and it is not clear how to prove such a result if one uses stochastic measure changes.

The manuscript is organized as follows. In Section 2 we gather the preliminary results. In Section 3 we discuss how importance sampling and measure changes can be carried out for MV-SDE, and in Section 4 we introduce our concept of optimality and identify the candidate optimal measure changes using the theory of large deviations. Section 5 illustrates numerically our results while proofs from Section 4 are carried out in Section 6.

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# 2. Preliminaries

Throughout the paper we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, where $\mathcal{F}_t$ is the augmented filtration of a standard multidimensional Brownian motion $W$.

We will work with $\mathbb{R}^d$, the $d$-dimensional Euclidean space of real numbers, and for $a = (a_1, \cdots, a_d) \in \mathbb{R}^d$ and $b = (b_1, \cdots, b_d) \in \mathbb{R}^d$ we denote by $|a|^2 = \sum_{i=1}^d a_i^2$ the usual Euclidean distance on $\mathbb{R}^d$ and by $\langle a, b\rangle = \sum_{i=1}^d a_i b_i$ the usual scalar product. For a $n \times l$-dimensional matrix $A \in \mathbb{R}^{n\times l}$ we denote by $A^\intercal$ its transpose and its Frobenius norm by $|A| = \mathrm{Tr}\{AA^\intercal\}^{1/2}$.

We consider some finite terminal time $T < \infty$ and use the following notation for spaces, which are standard in the McKean-Vlasov literature (see [8]). We define $\mathbb{S}^p$ for $p \geq 1$, as the space of $\mathbb{R}^d$-valued, $\mathcal{F}_\cdot$-adapted processes $Z$, that satisfy, $\mathbb{E}[\sup_{0\leq t\leq T} |Z_t|^p]^{1/p} < \infty$. Similarly, $L^p_t(\mathbb{R}^d)$, defines the space of $\mathbb{R}^d$-valued, $\mathcal{F}_t$-measurable random variables $X$, that satisfy, $\mathbb{E}[|X|^p]^{1/p} < \infty$. We define $L^2_0(\mathbb{R}^d)$ as the space of deterministic square-integrable functions $g : [0, T] \to \mathbb{R}^d$ with norm $\|g\|^2_{L^2_0} = \int_0^T |g(s)|^2 ds$; $C([0, T], \mathbb{R}^d)$ the space of continuous functions from $[0, T]$ to $\mathbb{R}^d$ and $C_0([0, T], \mathbb{R}^d)$ its sub-space of maps such that $f(0) = 0$.

Given the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, we denote by $\mathcal{P}(\mathbb{R}^d)$ the set of probability measures on this space, and write $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ if $\mu \in \mathcal{P}(\mathbb{R}^d)$ and for some $x \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} |x - y|^2 \mu(\mathrm{d}y) < \infty$. For $p \in \{1, 2\}$ we introduce following Wasserstein metrics on the space $\mathcal{P}_2(\mathbb{R}^d)$ for $\mu,\ \nu \in \mathcal{P}_2(\mathbb{R}^d)$ (see [15]),

$$W^{(p)}(\mu, \nu) = \inf_\pi \left\{ \left( \int_{\mathbb{R}^d\times\mathbb{R}^d} |x - y|^p \pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{p}} : \pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \text{ with marginals } \mu \text{ and } \nu \right\} .$$

The symbol $C$ is used throughout this work to denote a generic non-negative constant independent of the relevant parameters and may take different values at each occurrence.

### 2.1. McKean-Vlasov stochastic differential equations

Let $W$ be an $l$-dimensional Brownian motion, take $\sigma \in \mathbb{R}^{d\times l}$ and let $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ be a measurable map. In this paper we consider MV-SDEs written in the form,

$$\mathrm{d}X_t = b(t, X_t, \mu_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad X_0 = x_0, \tag{2.1}$$

where $\mu_t$ denotes the law of the process $X$ at time $t$, i.e. $\mu_t = \mathbb{P} \circ X_t^{-1}$. Consider the following assumption on the coefficients.

***Assumption* 2.1.** Let $\sigma \in \mathbb{R}^{d\times l}$ and assume that $b$ is jointly continuous (in $(t, x, \mu)$) and satisfies the following assumptions.

1. One-sided Lipschitz growth condition in $x$ and Lipschitz in law: there exists $L > 0$ such that for all $t \in [0, T]$, all $x, x' \in \mathbb{R}^d$ and all $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$ we have that
$$\langle x-x', b(t, x, \mu)-b(t, x', \mu)\rangle \le L|x-x'|^2 \quad \text{and} \quad |b(t, x, \mu)-b(t, x, \mu')| \le LW^{(2)}(\mu, \mu').$$
2. Locally Lipschitz with polynomial growth in $x$: there exists $q \in \mathbb{N}$ with $q > 1$ such that for all $t \in [0, T]$, all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and all $x,\ x' \in \mathbb{R}^d$ the following holds.
$$|b(t, x, \mu) - b(t, x', \mu)| \le L(1 + |x|^q + |x'|^q)|x - x'|.$$

Under these assumptions, existence and uniqueness follows from the results given in [15, Theorem 3.3].

**Theorem 2.2** ([15])**.** *Suppose that $b$ and $\sigma$ satisfy Assumption 2.1. Then there exists a unique solution for $X \in \mathbb{S}^m([0, T])$ to the MV-SDE (2.1). For any $m \ge 2$ there exists a positive constant $C$ such that*

$$\mathbb{E}\big[\sup_{t\in[0,T]} |X_t|^m\big] \le C\Big(|x_0|^m + \Big(\int_0^T |b(t, 0, \delta_0)|\mathrm{d}t\Big)^m + |\sigma|^m\Big)e^{CT}.$$

**The interacting particle system approximation and propagation of chaos** We approximate the equation (2.1) using an $N$-dimensional system of interacting particles $X^{i,N}$ satisfying the SDE

$$\mathrm{d}X_t^{i,N} = b\Big(t, X_t^{i,N}, \mu_t^N\Big)\mathrm{d}t + \sigma\mathrm{d}W_t^i, \quad \mu_t^N(\mathrm{d}x) := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}(\mathrm{d}x), \quad X_0^{i,N} = x_0, \tag{2.2}$$

where $\delta_{X_t^{j,N}}$ is the Dirac measure at point $X_t^{j,N}$, and the Brownian motions $\{W^i\}_{i\ge1}$ are independent from each other and from the BM $W$ appearing in (2.1). The term propagation of chaos refers to the convergence of the particle system to the original MV-SDE. Different formulations of this property exist; a common one is the pathwise convergence result where we consider the system of non interacting particles

$$\mathrm{d}X_t^i = b(t, X_t^i, \mu_t^{X^i})\mathrm{d}t + \sigma\mathrm{d}W_t^i, \quad X_0^i = x_0, \tag{2.3}$$

which are of course just MV-SDEs satisfying $\mu_t^{X^i} = \mu_t^X$ for all $i$. Under sufficient conditions, one can then prove the following convergence result (see [8]*Theorem 1.10 for example).

$$\lim_{N\to\infty} \sup_{1\le i\le N} \mathbb{E}\left[\sup_{0\le t\le T} |X_t^{i,N} - X_t^i|^2\right] = 0\,.$$

## 2.2. Large Deviation Principles

In this section, we state the main results from the large deviations theory that we use throughout the paper. For a full exposition the reader can consult texts such as [13] or [16]. The large deviation principle (LDP) characterizes the limiting behaviour in exponential scale, as $\epsilon \to 0$, of a family of probability measures $\{\mu_\epsilon\}$, defined on the space $(\mathcal{X}, \mathcal{B}_\mathcal{X})$, where $\mathcal{X}$ a topological space so that open and closed subsets of $\mathcal{X}$ are well-defined, and $\mathcal{B}_\mathcal{X}$ is the Borel $\sigma$-algebra on $\mathcal{X}$. The limiting behaviour is defined via a so-called rate function. We assume that the probability spaces have been completed, consequently, $\mathcal{B}_\mathcal{X}$ is a complete Borel $\sigma$-algebra on $\mathcal{X}$. We have the following definition [13, pg.4].

***Definition* 2.3** (Rate function)***.*** A rate function $I$ is a lower semicontinuous mapping $I : \mathcal{X} \to [0,\infty]$ (i.e. for all $\alpha \in [0,\infty)$, the sub-level set $\Psi_I(\alpha) := \{x : I(x) \le \alpha\}$ is a closed subset of $\mathcal{X}$). A good rate function is a rate function for which all the sub-level sets $\Psi_I(\alpha)$ are compact subsets of $\mathcal{X}$. The effective domain of $I$, denoted $D_I$, is the set of points in $\mathcal{X}$ of finite rate, namely, $D_I := \{x : I(x) < \infty\}$.

We use the standard notation: for any set $\Gamma$, $\overline{\Gamma}$ denotes the closure and $\Gamma^o$ denotes the interior of $\Gamma$. As is standard practice in LDP theory, the infimum of a function over an empty set is interpreted as $\infty$. We then define what it means for a sequence of measures to have an LDP [13]*pg.5.

***Definition* 2.4.** A family of probability measures, $\{\mu_\epsilon\}$ with $\epsilon > 0$ satisfies the large deviation principle with a rate function $I$ if, for all $\Gamma \in \mathcal{B}_\mathcal{X}$,

$$-\inf_{x\in\Gamma^o} I(x) \le \liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \le \limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \le -\inf_{x\in\overline{\Gamma}} I(x)\,. \tag{2.4}$$

It is also typical to have LDP defined in terms of a sequence of random variables $Z_\epsilon$, in which case one replaces $\mu_\epsilon(\Gamma)$ by $\mathbb{P}[Z_\epsilon \in \Gamma]$. The LDP for Brownian motion in path space is given by the celebrated Schilder's theorem, which states that for a $d$-dimensional Brownian motion $W$, the family $(\sqrt{\epsilon}W)_{\epsilon>0}$ satisfies an LDP with the good rate function (see [13])

$$I(y) = \begin{cases} \frac{1}{2}\int_0^T |\dot{y}_t|^2 \mathrm{d}t\,, & \text{if } y \in \mathbb{H}_T^d\,, \\ \infty, & \text{otherwise,} \end{cases}$$

where $\mathbb{H}_T^d$ denotes the space of $\mathbb{R}^d$-valued absolutely continuous functions with value 0 at 0 that possess a square integrable derivative.

The following result can be viewed as a generalisation of Laplace's approximation of integrals to the infinite dimensional setting and transfers the LDP from probabilities to expectations (see [13]).

***Lemma* 2.5** (Varadhan's Lemma). Let $Z_\epsilon$ be a family of random variables in $\mathcal{X}$ satisfying a large deviation principle with good rate function $I$ and let $\varphi : \mathcal{X} \to \mathbb{R}$ be a continuous function such that the following integrability (moments) condition holds for some $\gamma > 1$.

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{E}\left[\exp\left(\frac{\gamma}{\epsilon}\varphi(Z_\epsilon)\right)\right] < \infty\,.$$

Then,

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\left[\exp\left(\frac{1}{\epsilon}\varphi(Z_\epsilon)\right)\right] = \sup_{x\in\mathcal{X}} \big\{\varphi(x) - I(x)\big\}\,.$$

As is discussed in [25] and in the subsequent sections, one needs a slight extension to Varadhan's lemma which allows the function $\varphi$ to take the value $-\infty$. The extension is proved in [25].

***Lemma* 2.6.** Let $\varphi : \mathcal{X} \to [-\infty, \infty)$ and assume the conditions in Lemma 2.5 are satisfied. Then the following bounds hold for any $\Gamma \in \mathcal{B}_\mathcal{X}$

$$\sup_{x\in\Gamma^0} \{\varphi(x) - I(x)\} \leq \liminf_{\epsilon\to 0} \epsilon \log\left(\int_{\Gamma^o} \exp\left(\frac{1}{\epsilon}\varphi(Z_\epsilon)\right) \mathrm{d}\mu_\epsilon\right)$$
$$\leq \limsup_{\epsilon\to 0} \epsilon \log\left(\int_{\overline{\Gamma}} \exp\left(\frac{1}{\epsilon}\varphi(Z_\epsilon)\right) \mathrm{d}\mu_\epsilon\right) \leq \sup_{x\in\overline{\Gamma}}\{\varphi(x) - I(x)\}\,.$$

Unlike Varadhan's lemma, the previous lemma allows us to control the lim inf and lim sup even when they are not equal.

## 2.3. Importance Sampling and large deviations

To motivate our approach we recall ideas from the pioneering works [21], [25] and [31], which establish a connection between large deviations and importance sampling. Consider the problem of estimating $\mathbb{E}_\mathbb{P}[G(X)]$ where $X$ is a random variable/process. The standard Monte Carlo estimator of this quantity using $n$ samples writes,

$$\hat{\theta}^n := \frac{1}{n}\sum_{i=1}^{n} G(X^{(i)}),$$

where $X^{(i)}$, $i = 1, \dots, n$ are i.i.d. samples of $X$ under $\mathbb{P}$. Through Radon-Nikodym theorem we can rewrite this expectation under a new measure $\mathbb{Q}$ weighted by the Radon-Nikodym derivative, that is, $\mathbb{E}_{\mathbb{P}}[G(X)] = \mathbb{E}_{\mathbb{Q}}[G(X)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}]$. Although the two expectations are equal, the variance of the value under expectation under $\mathbb{Q}$ is,

$$\mathrm{Var}_{\mathbb{Q}}\Big[G(X)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\Big] = \mathbb{E}_{\mathbb{P}}\Big[G(X)^2\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\Big] - \mathbb{E}_{\mathbb{P}}\Big[G(X)\Big]^2. \tag{2.5}$$

Thus, we may be able to choose $\mathbb{Q}$, under which the *importance sampling estimator* defined by

$$\hat{\theta}^n_{\mathbb{Q}} := \frac{1}{n}\sum_{i=1}^{n} G(X^{(i)}_{\mathbb{Q}})\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)^{(i)},$$

where $X^{(i)}_{\mathbb{Q}}$ are i.i.d. samples of $X$ under $\mathbb{Q}$ and $\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)^{(i)}$ are the corresponding samples of the Radon-Nikodym density, will have a much smaller variance than the standard Monte Carlo estimator $\hat{\theta}^n$. As it turns out, if one can choose $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{G}{\mathbb{E}_{\mathbb{P}}[G]}$, then the variance of $\hat{\theta}^n_{\mathbb{Q}}$ under $\mathbb{Q}$ reduces to zero, i.e. we have no error in our Monte Carlo simulation. Unfortunately, in order to choose such a change of measure one would need to a priori know the value of $\mathbb{E}_{\mathbb{P}}[G(X)]$ i.e. the value we wish to estimate in the first place.

Instead one typically chooses $\mathbb{Q}$ to minimise (2.5) over a set of equivalent probability measures, chosen to add only a small amount of extra computation and such that the process $X$ is easy to simulate under the new measure. Specializing to the Brownian filtration, a common choice of $\mathbb{Q}$ is the Girsanov transform, (1.2) where $f$ is often taken to be a deterministic function.

For example in [32] the authors develop an importance sampling procedure in the context of Gaussian random vectors through a so-called "tilting" parameter, which corresponds to shifting the mean of the Gaussian random vector via a Girsanov transform. Although this method is intuitive, it still requires estimation of the Jacobian of $G$ w.r.t. the tilting parameter and applying Newton's method to select the optimal parameter value. These steps can be computationally expensive, and it is difficult to obtain rigorous optimality results.

Even after one has reduced the set of measures $\mathbb{Q}$ to optimise over, in general the problem of minimizing (2.5) will not have a closed form solution. To simplify the problem further, one can instead minimize a proxy for the variance obtained through Varadhan's lemma in the so-called small noise asymptotic regime as discussed in [21] and [25].

Assuming that a Girsanov change of measure is used and $G$ is non-negative (or bounded from below such that by the addition of a constant one can re-cast it as a non-negative function), we want to minimise

$$\mathbb{E}_{\mathbb{P}}\left[G(W)^2\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right] = \mathbb{E}_{\mathbb{P}}\left[\exp\left(2F(W) - \int_0^T f_t \mathrm{d}W_t + \frac{1}{2}\int_0^T f_t^2 \mathrm{d}t\right)\right], \tag{2.6}$$

where $F = \log(G)$. Recall that the extension to Varadhan's lemma we present allows for $G = 0$. Typically $G$ is defined as a functional of the SDE, but here with a slight abuse of notation we have redefined it as the functional of the driving Brownian motion. It is important for this type of argument that we are able to write the solution of the SDE in terms of BM as well, i.e. we can write $X_t = H(t, W_\cdot)$. Finding the optimal $f$ by minimizing (2.6) is in general intractable, hence an asymptotic approximation of the variance should be constructed. To this end, let us consider,

$$\epsilon \log \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{\epsilon}\left(2F(\sqrt{\epsilon}W) - \int_0^T \sqrt{\epsilon} f_t \mathrm{d}W_t + \frac{1}{2}\int_0^T f_t^2 \mathrm{d}t\right)\right)\right],$$

which equals log of (2.6) when $\epsilon = 1$. The *small noise asymptotic approximation* is then,

$$L(f) := \limsup_{\epsilon \to 0} \epsilon \log \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{\epsilon}\left(2F(\sqrt{\epsilon}W) - \int_0^T \sqrt{\epsilon} f_t \mathrm{d}W_t + \frac{1}{2}\int_0^T f_t^2 \mathrm{d}t\right)\right)\right].$$

One then computes a candidate variance reduction parameter $f^*$ by minimizing $L(f)$, which can be thought of as approximating $\mathbb{E}_{\mathbb{P}}\left[G(W)^2 \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right]$ by $\exp(L(f))$. Crucially, $L$ is in a form that can be evaluated using the Varadhan's lemma, i.e., we can change $L$ into a supremum depending on the rate function. The parameter $f^*$, which minimises $L$ over some predefined space is known as *asymptotically optimal*, see [25]. We will give a precise definition of this concept later. It is important to note that these approximations are not approximations for the original problem (calculate $\mathbb{E}_{\mathbb{P}}[G(X)]$), they are only used to choose the candidate measure change for variance reduction.

# 3. Importance sampling for MV-SDEs

Leaving LDPs and the optimality of the IS (importance sampling) on the side for a moment, let us discuss how IS can be implemented for MV-SDEs with a given measure change. Recall that MV-SDEs take the form (2.1). To make explicit the dependence of the law of the solution on the measure, we write $\mu^X_{t,\mathbb{P}} = \mathbb{P} \circ X_t^{-1}$, and we add the corresponding superscript to the Brownian motions, writing $W^{\mathbb{P}}$ instead of $W$ and $W^{i,\mathbb{P}}$ instead of $W^i$.

For the change of measure, one considers a Girsanov transform where the allowed functions are from the Cameron-Martin space of absolutely continuous functions with square integrable derivative, i.e.,

$$\mathbb{H}^d_T = \left\{h : [0,T] \mapsto \mathbb{R}^d :\ h_0 = 0\,,\ h_\cdot = \int_0^\cdot \dot{h}_t dt\,,\quad \int_0^T |\dot{h}_t|^2 \,\mathrm{d}t < \infty\right\}.$$

If $d = 1$ we just write $\mathbb{H}_T = \mathbb{H}^1_T$. For any $h \in \mathbb{H}^d_T$, we define an equilvalent probability measure $\mathbb{Q}$ as follows.

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T} = \mathcal{E}\left(\int_0^\cdot \langle \dot{h}_t, \mathrm{d}W^{\mathbb{P}}_t\rangle\right)_T = \exp\Big(\int_0^T \langle \dot{h}_t, \mathrm{d}W^{\mathbb{P}}_t\rangle - \frac{1}{2}\int_0^T |\dot{h}_t|^2 \,\mathrm{d}t\Big). \tag{3.1}$$

Under this new measure $\mathbb{Q}$, the process $W^{\mathbb{Q}}_{\cdot} = W^{\mathbb{P}}_{\cdot} - h_{\cdot}$ is a standard $d$-dimensional $\mathbb{Q}$-Brownian motion. We note that the Radon-Nikodym density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{E}(\int_0^{\cdot} \langle \dot{h}_s, \mathrm{d}W^{\mathbb{P}}_s \rangle)_t =: \mathcal{E}_t$ is itself the solution of the SDE

$$\mathrm{d}\mathcal{E}_t = \langle \dot{h}_t, \mathcal{E}_t \mathrm{d}W^{\mathbb{P}}_t \rangle, \quad \mathcal{E}_0 = 1 \quad \Rightarrow \quad \mathcal{E}_t = \exp\Big\{ \int_0^t \langle \dot{h}_s, \mathrm{d}W^{\mathbb{P}}_s \rangle - \frac{1}{2}\int_0^t |\dot{h}_s|^2 \mathrm{d}s \Big\}. \tag{3.2}$$

Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, one can also define $Z_t := \mathcal{E}_t^{-1} := \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}|_{\mathcal{F}_t}$. With our conditions on $\dot{h}$ it is also a straightforward task to show $\mathcal{E}_t$ and $Z_t$ are in $\mathbb{S}^p$ for all $p \geq 1$.

Recall our goal: *estimate* $\mathbb{E}_{\mathbb{P}}[G(X)] = \mathbb{E}_{\mathbb{Q}}[G(X)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}]$ *for a given pay-off functional* $G$ *by simulating* $X$ *under* $\mathbb{Q}$. In the following paragraphs we present two alternative approaches to achieve this goal. Both approaches are based on interacting particle systems, hence a propagation of chaos (PoC) result and a converging discretization scheme are required to justify the convergence of the Monte Carlo estimators. For the first approach (decoupling) such results are available in the literature, see e.g. [14] for a PoC and a Euler scheme (with rates) for super-linear growth MV-SDEs under conditions more general than Assumption (2.1). On the other hand, our second algorithm (full measure change) requires a PoC and a discretization scheme *after* the measure change. The former is a non-standard result and hence we provide one; the latter is more straightforward and can be adapted from the constructions in e.g. [14] so that we do not discuss it further.

## 3.1. A decoupling argument: fixing the empirical law

An obvious way to solve the problem of IS is to approximate the law of the MV-SDE under $\mathbb{P}$ and use that as a fixed input to a new equation which will be simulated under $\mathbb{Q}$. In this set-up the McKean-Vlasov SDE becomes an SDE with random coefficients. The algorithm is as follows.

1. Use (2.2) with $N$ particles to approximate (2.1) under $\mathbb{P}$. Use a numerical scheme (say Euler scheme as in [14]) to simulate the particles, approximating the empirical law $\mu^N_t$ over $[0,T]$.
   Define a new SDE, approximating the original MV-SDE (2.1), which is now a *standard SDE with random coefficients*
   $$\mathrm{d}\bar{X}_t = b(t, \bar{X}_t, \mu^N_t)\mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{P}}_t, \quad \bar{X}_0 = x_0, \tag{3.3}$$
   where $W^{\mathbb{P}}$ is a $\mathbb{P}$-Brownian motion independent of the $\{W^{i,\mathbb{P}}\}_{i=1,\cdots,N}$ appearing in (2.2). SDEs with random coefficients appear typically in optimal control, hence the reader can consult texts such as [33]*Chapter 1 for further details on existence and uniqueness for such SDEs.
2. Simulate (3.3) under the importance sampling measure $\mathbb{Q}$, i.e.,
   $$\mathrm{d}\bar{X}_t = \Big(b(t, \bar{X}_t, \mu^N_t) + \sigma \dot{h}_t\Big)\,\mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{Q}}_t, \quad \bar{X}_0 = x_0\,.$$
   This second run is therefore standard importance sampling, but the SDE has random coefficients i.e. the empirical law is random.

We will refer to algorithms of this form as *Decoupling Algorithms*. This scheme has the disadvantage that it requires an estimate of the error coming from the original approximation of the law and also that it requires more simulations; naively one can say twice the amount, but in effect the second simulation does not necessarily need to have the same sample size as the first one (we do not explore optimality in regard to this issue). Lastly, it is not a requirement to use interacting particles to approximate the law of the SDE, any approximation will work (quantization [23], Fokker-Plank methods [1], Cubature [11], Fourier techniques, to mention some). The goal here is to make the SDEs independent.

## 3.2. Complete Measure Change

An alternative is to change the measure under which we are simulating in the coefficients *and* the Brownian motion. This is not a simple problem and as far as we are aware changing the measure of a MV-SDE and its particle approximation is not discussed elsewhere in the literature for this purpose[1], we therefore provide a discussion along with the pitfalls here.

**The scalar interaction case.** Later in this section we present the *complete measure change* algorithm in the general setting of (2.1). However, this algorithm is more complex than the decoupled one and hence we choose to illustrate our ideas in a simplified setting first and return to the general case afterward. Consider the following MV-SDE.

$$\mathrm{d}X_t = \hat{b}\big(t, X_t, \mathbb{E}_{\mathbb{P}}[f(X_t)]\big)\mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{P}}, \quad X_0 = x_0\,, \quad t \in [0,T]\,. \tag{3.4}$$

where $\sigma \in \mathbb{R}^{d\times l}$ and assumptions on $f$ and $\hat{b}$ will be specified later. The measure changed version of (3.4) takes the following form, where again $Z := \mathcal{E}^{-1}$.

$$\begin{aligned}\mathrm{d}X_t &= \Big(\hat{b}(t, X_t, \mathbb{E}_{\mathbb{P}}[f(X_t)]) + \sigma \dot{h}_t\Big)\mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{Q}} \\ &= \Big(\hat{b}(t, X_t, \mathbb{E}_{\mathbb{Q}}\big[f(X_t)Z_t\big]) + \sigma \dot{h}_t\Big)\mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{Q}}.\end{aligned}$$

In view of simulation, we re-write the measure changed MV-SDE from (3.2) and (3.4) as a system

$$\mathrm{d}X_t = \Big(\hat{b}\Big(t, X_t, \mathbb{E}_{\mathbb{Q}}\big[f(X_t)Z_t\big]\Big) + \sigma \dot{h}_t\Big)\mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{Q}}, \quad \text{and} \quad \mathrm{d}Z_t = -Z_t \dot{h}_t \mathrm{d}W_t^{\mathbb{Q}}.$$

Note that although the couple $(X, Z)$ is still a MV-SDE in dimension $2d$, its coefficients may no longer be Lipschitz in the measure argument, whence the difficulty of proving a propagation of chaos result under the new measure.

We now write the interacting particle system for the pair $X, Z$ under $\mathbb{Q}$. For technical reasons which will become clear below, we replace $\hat{b}(t, x, y)$ with $\hat{b}_K(t, x, y) :=$

[1]Measures changes for MV-SDE appear in methods requiring to remove the drift altogether, for instance in establishing weak solutions to MV-SDEs, see e.g. [12].

$b(t,x,\Pi_K(y))$ where $\Pi_K$ is the projection operator onto the $d$-dimensional ball centered at the origin and with a radius $K=1+\|\mathbb{E}_{\mathbb{P}}[f(X_\cdot)]\|_\infty$.
The particle system takes the following form.

$$\mathrm{d}X_t^{i,N} = \Big(\hat{b}_K\big(t, X_t^{i,N}, \frac{1}{N}\sum_{j=1}^N f(X_t^{j,N})Z_t^{j,N}\big) + \sigma\dot{h}_t\Big)\mathrm{d}t + \sigma \mathrm{d}W_t^{i,\mathbb{Q}}, \quad X_0^{i,N} = x_0\,, \tag{3.5}$$

$$\mathrm{d}Z_t^{i,N} = -\langle \dot{h}_t, Z^{i,N}\mathrm{d}W_t^{i,\mathbb{Q}}\rangle, \quad Z_0^{i,N} = 1\,. \tag{3.6}$$

The importance sampling estimator of $\theta = \mathbb{E}_{\mathbb{P}}[G(X)]$ then takes the form

$$\hat{\theta}_h = \frac{1}{N}\sum_{i=1}^N Z_T^{i,N} G(X^{i,N}). \tag{3.7}$$

***Remark* 3.1.** One may be tempted to write the interacting particle approximation under $\mathbb{P}$,

$$\mathrm{d}X_t^{i,N} = \hat{b}_K\Big(t, X_t^{i,N}, \frac{1}{N}\sum_{j=1}^N f(X_t^{j,N})\Big)\mathrm{d}t + \sigma\mathrm{d}W_t^{i,\mathbb{P}},$$

and then change the measure for the particle system, writing

$$\mathrm{d}X_t^{i,N} = \Big(\hat{b}_K\big(t, X_t^{i,N}, \frac{1}{N}\sum_{j=1}^N f(X_t^{j,N})\big) + \sigma\dot{h}_t\Big)\mathrm{d}t + \sigma\mathrm{d}W_t^{i,\mathbb{Q}},$$

where we have taken the same $\dot{h}$ for every Brownian motion in order for all particles to have the same law. However, it is easy to see by the standard propagation of chaos result that as $N\to\infty$, this particle system converges to the solution of the MV-SDE

$$\mathrm{d}X_t = \Big(\hat{b}_K\big(t, X_t, \mathbb{E}_{\mathbb{Q}}[f(X_t)]\big) + \sigma\dot{h}_t\Big)\mathrm{d}t + \sigma\mathrm{d}W_t^{\mathbb{Q}} = \hat{b}_K\big(t, X_t, \mathbb{E}_{\mathbb{Q}}[f(X_t)]\big)\mathrm{d}t + \sigma\mathrm{d}W_t^{\mathbb{P}},$$

which is not what one is looking for.

### *3.2.1. The Propagation of chaos result for scalar interactions*

To state a propagation of chaos result for the particle system (3.5) we introduce the auxiliary system of non-interacting particles, (in fact (3.6) above and (3.9) below are the same).

$$\mathrm{d}X_t^i = \Big(\hat{b}\Big(t, X_t^i, \mathbb{E}_{\mathbb{Q}}\big[f(X_t^i)Z_t^i\big]\Big) + \sigma\dot{h}_t\Big)\mathrm{d}t + \sigma\mathrm{d}W_t^{i,\mathbb{Q}}\,, \quad X_0^i = x_0\,, \tag{3.8}$$

$$\mathrm{d}Z_t^i = -\langle \dot{h}_t, Z_t^i \mathrm{d}W_t^{i,\mathbb{Q}}\rangle, \quad Z_0^i = 1\,. \tag{3.9}$$

***Proposition* 3.1.** Let $\hat{b} : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$ be jointly continuous in its variables and assume that $x \mapsto \hat{b}(\cdot, x, \cdot)$ satisfies the one-sided Lipschitz and local Lipschitz/polynomial growth condition of Assumption 2.1 uniformly on the other variables, $y \mapsto \hat{b}(\cdot,\cdot,y)$ is Lipschitz uniformly on the other variables, $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ is Lipschitz; $\dot{h} \in L_0^2(\mathbb{R}^l)$. Then the system (3.8)–(3.9) has a unique strong solution. The same holds for equation (3.4).

Now, let $\hat{b}_K(t,x,y) := b(t,x,\Pi_K(y))$ where $\Pi_K$ is the projection operator onto the $d$-dimensional ball centered at the origin and with a radius $K = 1+\|\mathbb{E}_{\mathbb{P}}[f(X_\cdot)]\|_\infty$. Then for system (3.5)–(3.6) there exists a unique strong solution such that for any $p \geq 2$ we have $\sup_{N\geq 1}\sup_{1\leq i\leq N}\mathbb{E}_{\mathbb{Q}}[\sup_{0\leq t\leq T}|X_t^{i,N}|^p] < \infty$.

Moreover, the following pathwise propagation of chaos result holds,

$$\lim_{N\to\infty}\sup_{1\leq i\leq N}\mathbb{E}_{\mathbb{Q}}\left[\sup_{0\leq t\leq T}|X_t^{i,N} - X_t^i|^2\right] = 0,$$

with a convergence rate $O(1/\sqrt{N})$.

This proposition may be used to analyze the convergence of the Monte Carlo estimator (3.7). Indeed, due to the fact that there is no coupling (or law dependency) in $Z^{i,N}$, we have $Z^{i,N} = Z^i$ and then $\hat{\theta}_h$ can be represented as follows

$$\hat{\theta}_h = \frac{1}{N}\sum_{i=1}^N Z_T^i G(X^i) + \frac{1}{N}\sum_{i=1}^N Z_T^i\big(G(X^{i,N}) - G(X^i)\big).$$

The first term above converges to $\theta$ as $N \to \infty$ by the law of large numbers, and the second term can be shown, e.g., to converge to zero in probability using Proposition 3.1 if $G$ is sufficiently regular.

***Proof of Proposition 3.1.*** The idea of the proof is to appeal to the standard Grönwall type inequality, but this is made difficult due to the presence of $Z$ term in (3.8).

Due to the assumptions on the coefficients we have the following, by Theorem 2.2, the solution to (3.4) exists, is unique and all its $p$-moments exist for $p \geq 2$ (i.e. belongs to $\mathbb{S}^p(\mathbb{P})$ $p \geq 2$). Via continuity and integrability of the solution of (3.4) and the properties of $f$, the map $[0,T] \ni t \mapsto \mathbb{E}_{\mathbb{P}}[f(X_t)]$ is uniformly bounded, hence by uniqueness it is obvious that replacing $\hat{b}$ with $\hat{b}_K$ in (3.4) does not change the solution. This, together with the Girsanov theorem, ensures that the system (3.8)-(3.9) admits a unique strong solution belonging to $\mathbb{S}^p(\mathbb{Q})$ for all $p \geq 2$. We shall then move on to the study of Equation (3.5).

**Existence and uniqueness of the solution of** (3.5) To show existence, consider the modified system

$$\mathrm{d}\overline{X}_t^{i,N} = \Big(\hat{b}_K\big(t, \overline{X}_t^{i,N}, \frac{1}{N}\sum_{j=1}^N f(\overline{X}_t^{j,N})\overline{Z}_t^{j,N}\big) + \sigma\dot{h}_t\Big)\mathrm{d}t + \sigma\mathrm{d}W_t^{i,\mathbb{Q}}, \quad \overline{X}_0^{i,N} = x_0\,,$$

$$\mathrm{d}\overline{Z}_t^{i,N} = -\phi_M(\overline{Z}_t^{1,N},\ldots,\overline{Z}_t^{N,N})\langle\dot{h}_t, \overline{Z}_t^{i,N}\mathrm{d}W_t^{i,\mathbb{Q}}\rangle, \quad \overline{Z}_0^{i,N} = 1\,,$$

where $\phi_M : \mathbb{R}^N \to [0,1]$ is a smooth compactly supported function such that $\phi_M = 1$ on $[0,C]^M$. It is easy to see that this system satisfies the assumption of Theorem II.3.6 in [29] and therefore admits a unique strong solution. This solution coincides with the solution of the original system on $[0,\tau^M]$, where $\tau_M = \inf\{t \geq 0 : (Z^{1,N},\dots,Z^{N,N}) \notin [0,C]^M\}$. The solution of the original system is thus also unique on this interval. Since $Z^{1,N},\dots,Z^{N,N}$ are strictly positive and continuous, $\tau_M \to \infty$ a.s. as $M \to \infty$, and a global solution of the original system can be constructed by gluing together the solutions of the modified system obtained with increasing values of $M$, and, similarly, uniqueness can be extended to all $t$ by iterating over $M$.

**Moment bounds for the solution of** (3.5) Critically, this would be much harder to show for (3.5) using $\hat{b}$ instead of $\hat{b}_K$, but the truncation allows us to obtain bounds on moments of all orders of the solution that are uniform in $N$ (note that $y \mapsto \hat{b}_K(\cdot,\cdot,y)$ is uniformly Lipschitz). To prove the uniform moment bound one relies on a much simplified version of the arguments used below to prove the PoC (see also the proof of Step 3 of Lemma 6.2). In rough, we apply Itô's formula to $|X^{i,N}|^2$ of (3.8), add and subtract $\hat{b}_K\big(\cdot,0,\frac{1}{N}\sum_{j=1}^N f(X^{j,N}_\cdot)Z^{j,N}_\cdot\big)$ to the inner product, use the one-sided Lipschitz condition, the uniform boundedness of $(t,y) \mapsto \hat{b}_K(t,0,y)$ and Young's inequality. This leads to the following estimate:

$$|X_t^{i,N}|^2 \leq C\int_0^t |X_s^{i,N}|^2 ds + x_0^2 + \sigma^2 t + 2\int_0^t \langle X_s^{i,N}, \sigma dW_s\rangle,$$

for some $C < \infty$. Now raising both sides to the power $p/2$, taking $\mathbb{Q}$-expectation and applying BDG inequality to the stochastic integral, we are in position to use Grönwall's inequality to obtain the uniform moment bounds.

**Propagation of chaos** Let $t \in [0,T]$, $i = 1,\cdots,N$, then Itô's lemma yields,

$$\begin{aligned}&|X_t^{i,N} - X_t^i|^2 \\ &= 2\int_0^t \Big\langle X_s^{i,N} - X_s^i, \hat{b}_K\big(s, X_s^{i,N}, \frac{1}{N}\sum_{j=1}^N f(X_s^{j,N})Z_s^j\big) - \hat{b}_K\big(s, X_s^i, \mathbb{E}_{\mathbb{Q}}[f(X_s^i)Z_s^i]\big)\Big\rangle \mathrm{d}s\,,\end{aligned} \tag{3.10}$$

where we used that $\hat{b}\big(\cdot,\cdot,\mathbb{E}_{\mathbb{Q}}[f(X_\cdot^i)Z_\cdot^i]\big) = \hat{b}_K\big(\cdot,\cdot,\mathbb{E}_{\mathbb{Q}}[f(X_\cdot^i)Z_\cdot^i]\big)$.

Let $s \in [0,T]$ and introduce in $\langle,\rangle$ the terms, $\hat{b}_K\big(s, X_s^i, \frac{1}{N}\sum_{j=1}^N f(X_s^{j,N})Z_s^j\big)$ and $\hat{b}_K\big(s, X_s^i, \frac{1}{N}\sum_{j=1}^N f(X_s^j)Z_s^j\big)$, where the empirical measure in the second term is the one constructed from the i.i.d. SDEs in (3.8), hence each $X^j$ corresponds to an independent realization of the MV-SDE. Splitting the integrand in (3.10) in three parts and using Cauchy-Schwarz and Young's inequality together with the conditions on $f$ and $b$ we obtain

$$\Big\langle X_s^{i,N}-X_s^i, \hat b_K\big(s,X_s^{i,N},\frac{1}{N}\sum_{j=1}^N f(X_s^{j,N})Z_s^j\big)-\hat b_K\big(s,X_s^i,\frac{1}{N}\sum_{j=1}^N f(X_s^{j,N})Z_s^j\big)\Big\rangle \le C|X_s^{i,N}-X_s^i|^2,$$

$$\Big\langle X_s^{i,N}-X_s^i, \hat b_K\big(s,X_s^i,\frac{1}{N}\sum_{j=1}^N f(X_s^j)Z_s^j\big)-\hat b_K\big(s,X_s^i,\mathbb{E}_{\mathbb{Q}}[f(X_s^i)Z_s^i]\big)\Big\rangle$$
$$\le C|X_s^{i,N}-X_s^i|^2 + C|\frac{1}{N}\sum_{j=1}^N f(X_s^j)Z_s^j-\mathbb{E}_{\mathbb{Q}}[f(X_s^i)Z_s^i]|^2,$$

and

$$\Big\langle X_s^{i,N}-X_s^i, \hat b_K\big(s,X_s^i,\frac{1}{N}\sum_{j=1}^N f(X_s^{j,N})Z_s^j\big)-\hat b_K\big(s,X_s^i,\frac{1}{N}\sum_{j=1}^N f(X_s^j)Z_s^j\big)\Big\rangle$$
$$\le \frac{C}{N}|X_s^{i,N}-X_s^i|\sum_{j=1}^N |X_s^{j,N}-X_s^j|Z_s^j.$$

Plugging all these together we obtain the following bound

$$\mathbb{E}_{\mathbb{Q}}\Big[\sup_{0\le t\le T}|X_t^{i,N}-X_t^i|^2\Big] \le C\int_0^T \mathbb{E}_{\mathbb{Q}}[|X_s^{i,N}-X_s^i|^2]+\mathbb{E}_{\mathbb{Q}}\Big[|X_s^{i,N}-X_s^i|\frac{1}{N}\sum_{j=1}^N|X_s^{j,N}-X_s^j|Z_s^j\Big]$$
$$+\mathbb{E}_{\mathbb{Q}}\Big[\big|\frac{1}{N}\sum_{j=1}^N f(X_s^j)Z_s^j-\mathbb{E}_{\mathbb{Q}}[f(X_s^i)Z_s^i]\big|^2\Big]\mathrm{d}s\,.$$

Concentrating on the second term of the three and using Young's inequality we have

$$\mathbb{E}_{\mathbb{Q}}\Big[|X_s^{i,N}-X_s^i|\frac{1}{N}\sum_{j=1}^N|X_s^{j,N}-X_s^j|Z_s^j\Big]$$
$$\le \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\Big[|X_s^{i,N}-X_s^i|^2\frac{1}{N}\sum_{j=1}^N\big(Z_s^j\big)^2\Big]+\frac{1}{2}\mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{N}\sum_{j=1}^N|X_s^{j,N}-X_s^j|^2\Big],$$
$$=\frac{1}{2}\mathbb{E}_{\mathbb{Q}}\Big[|X_s^{i,N}-X_s^i|^2\frac{1}{N}\sum_{j=1}^N\big(Z_s^j\big)^2\Big]+\frac{1}{2}\mathbb{E}_{\mathbb{Q}}\Big[|X_s^{i,N}-X_s^i|^2\Big],$$

since all particles are identically distributed. Recalling that for a product of random variables $X$, $Y$ one has $\mathbb{E}[XY]=\mathbb{E}[X]\mathbb{E}[Y]+\mathrm{Cov}(X,Y)$,

$$\mathbb{E}_{\mathbb{Q}}\Big[|X_s^{i,N}-X_s^i|^2\frac{1}{N}\sum_{j=1}^N\big(Z_s^j\big)^2\Big]$$
$$=\mathbb{E}_{\mathbb{Q}}\Big[|X_s^{i,N}-X_s^i|^2\Big]\mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{N}\sum_{j=1}^N\big(Z_s^j\big)^2\Big]+\mathrm{Cov}_{\mathbb{Q}}\Big(|X_s^{i,N}-X_s^i|^2,\frac{1}{N}\sum_{j=1}^N\big(Z_s^j\big)^2\Big).$$

Using the integrability of $Z$ we can bound the second factor in the 1st term by a constant. Further, using that the $Z^i$'s are i.i.d., the definition of covariance and applying Cauchy-Schwarz inequality, we obtain,

$$\Big|\mathrm{Cov}_{\mathbb{Q}}\Big(|X_s^{i,N}-X_s^i|^2,\frac{1}{N}\sum_{j=1}^N\big(Z_s^j\big)^2\Big)\Big|\leq\mathrm{Var}_{\mathbb{Q}}\Big(|X_s^{i,N}-X_s^i|^2\Big)^{1/2}\mathrm{Var}_{\mathbb{Q}}\Big(\frac{1}{N}\sum_{j=1}^N\big(Z_s^j\big)^2\Big)^{1/2}$$
$$=\mathrm{Var}_{\mathbb{Q}}\Big(|X_s^{i,N}-X_s^i|^2\Big)^{1/2}\frac{1}{\sqrt{N}}\mathrm{Var}_{\mathbb{Q}}\Big(\big(Z_s^i\big)^2\Big)^{1/2}\leq\frac{C}{\sqrt{N}}$$

where we used the fact that the fourth moment of $X^i$ is bounded and fourth moment of $X^{i,N}$ is bounded uniformly on $N$ (as shown at the beginning of the proof).

Combining this with our previous bounds yields,

$$\mathbb{E}_{\mathbb{Q}}\Big[\sup_{0\leq t\leq T}|X_t^{i,N}-X_t^i|^2\Big]$$
$$\leq C\int_0^T\mathbb{E}_{\mathbb{Q}}[|X_s^{i,N}-X_s^i|^2]+N^{-1/2}+\mathbb{E}_{\mathbb{Q}}\Big[\Big|\frac{1}{N}\sum_{j=1}^N f(X_s^j)Z_s^j-\mathbb{E}_{\mathbb{Q}}[f(X_s^i)Z_s^i]\Big|^2\Big]\mathrm{d}s\,.$$

Finally, taking supremum over $i$, using Grönwall's lemma, the fact that the $X^j$'s and $Z^j$'s are i.i.d. with bounded fourth moments, and that $f$ is Lipschitz we obtain,

$$\sup_{1\leq i\leq N}\mathbb{E}_{\mathbb{Q}}\Big[\sup_{0\leq t\leq T}|X_t^{i,N}-X_t^i|^2\Big]\leq Ce^C\int_0^T\frac{1}{\sqrt{N}}+\mathbb{E}_{\mathbb{Q}}\Big[\Big|\frac{1}{N}\sum_{j=1}^N f(X_s^j)Z_s^j-\mathbb{E}_{\mathbb{Q}}[f(X_s^i)Z_s^i]\Big|^2\Big]\mathrm{d}s$$
$$\leq\frac{C}{\sqrt{N}}\int_0^T 1+\mathbb{E}_{\mathbb{Q}}\Big[\big|f(X_s^1)Z_s^1-\mathbb{E}_{\mathbb{Q}}[f(X_s^1)Z_s^1]\big|^2\Big]\mathrm{d}s\ \to 0,$$

where in the last inequality we use CLT and take $N\to\infty$, which concludes the proof. □

***Remark* 3.2** (More general $\sigma$ - time-space dependency)*.* A careful inspection of our computations shows that when $h$ is deterministic, the above arguments along with standard ones for a non measure-changed propagation of chaos proof (e.g. [14]*Proposition 3.1) allow one to extend this result to the case when $\sigma$ is Lipschitz in space (and time-dependent). It is not clear how to prove a propagation of chaos result when $\sigma$ also depends on the measure.

*3.2.2. A second PoC with general measure dependency*

In this section we prove another propagation of chaos result under the measure change. The scope of this result is a tradeoff between the generality of the measure dependence and the structural assumption of Proposition 3.1. We assume that $b$ is uniformly bounded and that $\mu \mapsto b(\cdot,\cdot,\mu)$ is $W^{(1)}$-Lipschitz (note that $W^{(1)}$-Lipschitz implies $W^{(2)}$-Lipschitz). Many aspects of the proof below are close to the proof of Proposition 3.1. The measure changed version of (2.1) takes the form,

$$\mathrm{d}X_t = \Big(b(t, X_t, \mu^X_{t,\mathbb{P}}) + \sigma \dot{h}_t\Big)\mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{Q}}_t, \quad \mu^X_{t,\mathbb{P}}(\cdot) = \int_{\mathbb{R}} z\mu^{(X,Z)}_{t,\mathbb{Q}}(\cdot, \mathrm{d}z), \tag{3.11}$$

where $Z$ is defined by (3.2), and $\mu^{(X,Z)}_{t,\mathbb{Q}} = \mathbb{Q} \circ (X_t, Z_t)^{-1}$. As before we replace $b(t, x, \mu)$ with $b_K(t, x, \mu) := \Pi_K(b(t, x, \mu))$ where $\Pi_K$ is the projection operator onto the $d$-dimensional ball centered at the origin and with a radius $K = \|b\|_\infty$.

Given the uniqueness of the solution to (2.1), it is clear that (3.11) has the same solution with either $b$ or with $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (t, x, \mu) \mapsto b_K(t, x, \mu) := \Pi_K(b(t, x, \mu))$; observe that $\Pi_K$ is a projection operator and hence has bounded norm (and is Lipschitz in Euclidean distance).

As above, we introduce the interacting particle system $(X^{i,N}, Z^i)_{i=1,\dots,N}$ with $Z^i$ given by (3.9) and $X^{i,N}$ defined as follows

$$\mathrm{d}X^{i,N}_t = \Big(b_K\big(t, X^{i,N}_t, \hat{\mu}^{X,N}_t\big) + \sigma \dot{h}_t\Big)\mathrm{d}t + \sigma \mathrm{d}W^{i,\mathbb{Q}}_t, \text{ with } \hat{\mu}^{X,N}_t(\mathrm{d}x) = \frac{1}{N}\sum_{j=1}^N Z^j_t \delta_{X^{j,N}_t}(\mathrm{d}x). \tag{3.12}$$

In the non-interacting particle system $(X^i, Z^i)_{i=1,\dots,N}$, $Z^i$ is once again defined by (3.9) and the dynamics of $X^i$ is as follows (compare with (3.11)), again using $b$ or $b_K$ as drift function leads to the same result,

$$\mathrm{d}X^i_t = \Big(b_K\Big(t, X^i_t, \int_{\mathbb{R}^d} z\mu^{(X,Z)}_{t,\mathbb{Q}}(\cdot, \mathrm{d}z)\Big) + \sigma \dot{h}_t\Big)\mathrm{d}t + \sigma \mathrm{d}W^{i,\mathbb{Q}}_t, \tag{3.13}$$

where the joint law $\mu^{(X^i,Z^i)}_{t,\mathbb{Q}} = \mathbb{Q} \circ (X^i_t, Z^i_t)^{-1}$ coincides with $\mathbb{Q} \circ (X_t, Z_t)^{-1} = \mu^{(X,Z)}_{t,\mathbb{Q}}$ since the variables $(X^i, Z^i)$'s are i.i.d.

**Theorem 3.2.** *Let Assumption 2.1 hold. Additionally assume that $\|b\|_\infty < \infty$ ($b$ is uniformly bounded); that $\mu \mapsto b(\cdot,\cdot,\mu)$ satisfies a Lipschitz condition also in the $W^{(1)}$-metric (uniformly wrt $t, x$); and there exists some $L > 0$ such that for any $t, x, y$ and any finite positive measure*[2] *$\vartheta$ on $\mathbb{R}^d$ we have*

$$|b(t, x, \vartheta) - b(t, y, \vartheta)| \le L|x - y|\big(1 + \vartheta(\mathbb{R}^d)\big). \tag{3.14}$$

[2] A measure $\vartheta$ is said to be positive if for any non-negative $f$, one has $\int f d\vartheta \ge 0$.

*Let $X^{i,N}$ denote the particle system (3.12) approximating $X^i_t$ defined in (3.13) ($Z^i$ defined by (3.9)).*

*Then the conclusion of Proposition 3.1 holds for these processes, i.e., the system (3.12), (3.6) has a unique strong solution satisfying* $\sup_{N\geq 1}\sup_{1\leq i\leq N}\mathbb{E}_{\mathbb{Q}}[\sup_{0\leq t\leq T}|X^{i,N}_t|^p]<\infty$ *for any $p\geq 2$.*

*Moreover, the following pathwise propagation of chaos result holds,*

$$\lim_{N\to\infty}\sup_{1\leq i\leq N}\mathbb{E}_{\mathbb{Q}}\left[\sup_{0\leq t\leq T}|X^{i,N}_t-X^i_t|^2\right]=0,$$

*with a convergence rate $O(1/\sqrt{N})$ for $d<4$, $O((\log N)/\sqrt{N})$ for $d=4$ and $O(N^{-2/d})$ if $d>4$.*

We point out that the convergence rate at higher dimensions is not optimal.

***Remark* 3.3** (On the assumptions)**.** It is the uniform boundedness of $b$, enforced via $b_K$, that yields the boundedness of the moments of the particle system uniformly in the number of particles. As in the proof of Proposition 3.1 this uniform boundedness of the moments is crucial in showing the propagation of chaos. Additionally, and for this general measure dependency, since the boundedness of $b$ needs to enforced via $b_K$ the one-sided Lipschitz structure is lost. Remark 3.4 addresses SDEs with drifts that are a combination of Proposition 3.1 and Theorem 3.2.

The assumptions cover the linear interaction case i.e. $b(t,x,\mu)=\int_{\mathbb{R}^d}\hat{b}(t,x,y)\mu(dy)$ for some uniformly bounded Lipschitz function $\hat{b}$ (see Section 5 for a particular example); and also the convolution case, i.e. $b(t,x,\mu)=\int_{\mathbb{R}^d}\tilde{b}(x-y)\mu(dy)$ for some uniformly bounded Lipschitz function $\tilde{b}$. If one has this example in mind, then condition (3.14) becomes more obvious. When $\mu$ is a probability measure then (3.14) is just a standard Lipschitz assumption, but during the proof, $\mu$ is replaced by the random empirical distribution $\hat{\mu}^{X,N}$ in (3.12), which due to its weights $(Z^i)_i$ makes the analysis more involved.

***Proof.*** Let $t\in[0,T]$, $i=1,\cdots,N$. We introduce the empirical law $\nu_t^{(X,Z),N}$ to approximate $\mu_{t,\mathbb{Q}}^{(X,Z)}$.

Since the variables $(X^i,Z^i)$ are i.i.d., we know (see [9]*Theorem 5.8) that the following convergence holds in $W^{(2)}$-metric,

$$\nu_t^{(X,Z),N}(\mathrm{d}x,\mathrm{d}z):=\frac{1}{N}\sum_{j=1}^N\delta_{X^j_t}(\mathrm{d}x)\delta_{Z^j_t}(\mathrm{d}z)\overset{N\to\infty}{\longrightarrow}\mu_{t,\mathbb{Q}}^{(X^i,Z^i)}(\mathrm{d}x,\mathrm{d}z)=\mu_{t,\mathbb{Q}}^{(X,Z)}(\mathrm{d}x,\mathrm{d}z).$$

**Existence, uniqueness and moment bounds**. These results are straightforward in view of the arguments already used in the proof of Proposition 3.1; the localization argument over the $Z^i$'s is crucial to make (3.14) a true Lipschitz condition (with $\vartheta$ replaced by $\hat{\mu}^{X,N}$). The finite moment bounds argument (uniform in the particles) is as in the proof of Proposition 3.1 once one observes that $|b_K|\leq\|b\|_\infty$ by construction.

**The Propagation of Chaos.** As in the proof of Proposition 3.1, Itô formula yields

$$|X_t^{i,N} - X_t^i|^2 = 2\int_0^t \Big\langle X_s^{i,N} - X_s^i, b\big(s, X_s^{i,N}, \hat\mu_s^{X,N}\big) - b\big(s, X_s^i, \int_{\mathbb{R}} z\mu_{s,\mathbb{Q}}^{(X,Z)}(\cdot, \mathrm{d}z)\big)\Big\rangle \mathrm{d}s\,. \tag{3.15}$$

We proceed as in the previous proof and add and subtract in $\langle , \rangle$ the terms $b(s, X_s^i, \hat\mu_s^{X,N})$ and $b(s, X_s^i, \int_{\mathbb{R}} z\nu_s^{(X,Z),N}(\cdot, \mathrm{d}z))$. For the 1st term we use (3.14) and that $\hat\mu_s^{X,N}$ is a positive measure, while for the second one we use the $W^{(1)}$-Lipschitz property

$$\begin{aligned}\Big\langle X_s^{i,N} - X_s^i, b\big(s, X_s^{i,N}, \hat\mu_s^{X,N}\big) - b\big(s, X_s^i, \hat\mu_s^{X,N}\big)\Big\rangle &\\ &\le C|X_s^{i,N} - X_s^i| \times |X_s^{i,N} - X_s^i| \times (1 + \hat\mu_s^{X,N}(\mathbb{R}^d))\\ &\le C|X_s^{i,N} - X_s^i|^2 + C|X_s^{i,N} - X_s^i|^2 \times \frac{1}{N}\sum_{j=1}^N Z_s^j\end{aligned}$$

and

$$\begin{aligned}&\Big\langle X_s^{i,N} - X_s^i, b\big(s, X_s^i, \hat\mu_s^{X,N}\big) - b\big(s, X_s^i, \int_{\mathbb{R}} z\nu_s^{(X,Z),N}(\cdot, \mathrm{d}z)\big)\Big\rangle\\ &\quad\le C|X_s^{i,N} - X_s^i| \times W^{(1)}\Big(\int_{\mathbb{R}} z\mu_s^{(X,Z),N}(\cdot, \mathrm{d}z), \int_{\mathbb{R}} z\nu_s^{(X,Z),N}(\cdot, \mathrm{d}z)\Big)\\ &\quad\le C|X_s^{i,N} - X_s^i|\frac{1}{N}\sum_{j=1}^N |X_s^{j,N} - X_s^j|(Z_s^j)^2,\\ &\quad\le C|X_s^{i,N} - X_s^i|^2\frac{1}{N}\sum_{j=1}^N (Z_s^j)^4 + C\frac{1}{N}\sum_{j=1}^N |X_s^{j,N} - X_s^j|^2,\end{aligned}$$

where for in the last two lines we dominated the Wasserstein distance by the moment difference and used $ab \le a^2 + b^2$. For the last difference, we have the following estimate.

$$\begin{aligned}&\Big\langle X_s^{i,N} - X_s^i, b\big(s, X_s^i, \int_{\mathbb{R}} z\nu_s^{(X,Z),N}(\cdot, \mathrm{d}z)\big) - b\big(s, X_s^i, \int_{\mathbb{R}} z\mu_{s,\mathbb{Q}}^{(X,Z)}(\cdot, \mathrm{d}z)\big)\Big\rangle\\ &\qquad\le C|X_s^{i,N} - X_s^i|^2 + CW^{(2)}\Big(\int_{\mathbb{R}} z\nu_s^{(X,Z),N}(\cdot, \mathrm{d}z), \int_{\mathbb{R}} z\mu_{s,\mathbb{Q}}^{(X,Z)}(\cdot, \mathrm{d}z)\Big)^2.\end{aligned}$$

Then following the steps of the preceding proof: joining all the estimates, taking $\sup_t$, $\mathbb{E}_{\mathbb{Q}}[\cdot]$, dealing with the integrand terms as before where one now sees a term $\mathrm{Var}_{\mathbb{Q}}\big(\frac{1}{N}\sum_{j=1}^N Z_s^j\big)$ and a term $\mathrm{Var}_{\mathbb{Q}}\big(\frac{1}{N}\sum_{j=1}^N \big(Z_s^j\big)^4\big)$ appearing instead of the term $\mathrm{Var}_{\mathbb{Q}}\big(\frac{1}{N}\sum_{j=1}^N \big(Z_s^j\big)^2\big)$, taking supremum over $i$ and using Grönwall's lemma, we have

$$\begin{aligned}&\sup_{1\le i\le N} \mathbb{E}_{\mathbb{Q}}\Big[\sup_{0\le t\le T} |X_t^{i,N} - X_t^i|^2\Big]\\ &\qquad\le Ce^C \int_0^T \frac{1}{\sqrt{N}} + \mathbb{E}_{\mathbb{Q}}\Big[W^{(2)}\Big(\int_{\mathbb{R}} z\nu_s^{(X,Z),N}(\cdot, \mathrm{d}z), \int_{\mathbb{R}} z\mu_{s,\mathbb{Q}}^{(X,Z)}(\cdot, \mathrm{d}z)\Big)^2\Big]\mathrm{d}s \overset{N\to\infty}{\longrightarrow} 0,\end{aligned}$$

where we use the [9]*Theorem 5.8 for the convergence of the $W^{(2)}$-distance since $X^i, Z^i \in \mathcal{S}^p$ for all $p \geq 2$. That results yields a dimension dependent convergence rate $O(1/\sqrt{N})$ for $d < 4$, $O((\log N)/\sqrt{N})$ for $d = 4$ and $O(N^{-2/d})$ if $d > 4$. □

***Remark* 3.4** (Combining Proposition 3.1 and Theorem 3.2)**.** By inspection of the proofs of Proposition 3.1 and Theorem 3.2 one can see that both settings can be combined to address general MV-SDEs with drifts given by linear combinations of the drifts in each of the results, e.g. $b(t,x,\mu) = x - x^3 + \int_{\mathbb{R}} \sin(x-y)\mu(dy)$.

*3.2.3. The Complete Measure Change Algorithm*

We now describe the algorithm for simulating a general MV-SDE under a complete measure change.

1. Simulate the $2d$-dimensional particle system for the MV-SDE after the measure change:

$$\begin{aligned} \mathrm{d}X_t^{i,N} &= \left( b\left( t, X_t^{i,N}, \frac{1}{N}\sum_{j=1}^N Z_t^j \delta_{X_t^{j,N}} \right) + \sigma \dot{h}_t \right) \mathrm{d}t + \sigma \mathrm{d}W_t^{i,\mathbb{Q}}, \quad X_0^{i,N} = x_0, \\ \mathrm{d}Z_t^i &= -\langle \dot{h}_t, Z_t^i \mathrm{d}W_t^{i,\mathbb{Q}} \rangle, \quad Z_0^i = 1\,. \end{aligned}$$

2. Compute the importance sampling estimator using the following formula:

$$\hat{\theta}_h = \frac{1}{N}\sum_{i=1}^N Z_T^{i,N} G(X^{i,N}).$$

We will refer to algorithms of this form as *Complete Measure Change Algorithms*. An advantage one can immediately see is that one simulates the particles only once. A key disadvantage is that the importance sampling to estimate the object of interest $\mathbb{E}[G(X)]$, may yield a poorer estimation of the original law $\mu$ and the term $\mathbb{E}_{\mathbb{Q}}[f(X_t)Z_t]$ in (3.8). We will discuss this in Section 5.

# 4. Optimal Importance Sampling for McKean-Vlasov SDEs

The previous section detailed algorithms for simulating MV-SDEs under an arbitrary change of measure. We now use the theory of large deviations to determine, in a certain optimal way, a measure change which will reduce the variance of the estimate.

An important point here is that we will be using the LDP for Brownian motion, rather than that for MV-SDEs. There are several works dealing with Large Deviations for MV-SDEs and their associated interacting particles systems, see [7], [19], [15] but such results are not of use here since we must be able to cheaply simulate the MV-SDE after the change of measure. We restrict to Girsanov measure changes since we know how the SDE changes under the measure change.

In this section we first show how the LDP framework can be applied to both algorithms to yield a simplified optimisation problem for finding the asymptotically optimal measure change (Theorems 4.5 and 4.6). In the last part we particularise our problem to the case where the functional to evaluate only depends on the terminal value of the MV-SDE ($G(X) = G(X_T)$) and demonstrate how the resulting simplified optimization problems can be solved in practice.

## 4.1. Preliminaries

We recall some of the main concepts for importance sampling with LDP, see [25] and [21] for further discussion. We denote by $\mathbb{W}^d_T$ the standard $d$-dimensional Wiener space of continuous functions over the time interval $[0,T]$ which are zero at time zero and in the one-dimensional case we simply write $\mathbb{W}_T$ instead of $\mathbb{W}^1_T$. This space is endowed with the topology of uniform convergence and with the usual Wiener measure $\mathbb{P}$, defined on the completed filtration $\mathcal{F}_T$, which makes the process $\mathbf{W}_t(x) = x_t$ with $x \in \mathbb{W}^d_T$ a standard $d$-dimensional Brownian motion. The goal is to estimate the expected value of some functional $\tilde{G} : \mathbb{W}^d_T \to \mathbb{R}_+$ continuous in the uniform topology, which is defined by the solution of the MV-SDE as functional of the Brownian trajectory (the precise definition of $\tilde{G}$ will be given later). We consider MV-SDEs of the form (2.1), and make the following assumptions.

***Assumption* 4.1.** Let Assumption 2.1 hold with $d = l$. Assume that $\sigma$ is non-degenerate (strict positive or negative definite matrix).

In view of Section 2, this assumption guarantees the existence of a unique strong solution to (2.1) with all moments. We further use the following assumption for the terminal function $G$. Note that this assumption is on $G$ as a function of the SDE, rather than the driving Brownian motion as is the case in [25]. Also, by addition of a constant, non-negativity of $G$ is ensured as long as it is bounded from below.

***Assumption* 4.2.** The functional $G$ is non-negative, continuous with respect to the sup norm and satisfies the following growth condition

$$G(x) \leq C_1 + C_2 \sup_{t\in[0,T]} |x_t|^p \,,$$

for $x : [0,T] \to \mathbb{R}^d$ a continuous function starting at $x_0$ where $C_1,\ C_2$ are positive constants and $p > 1$.

The notion of "optimality" for the measure change used is the so-called *asymptotic optimality*, as defined in[3] [21]. From the approach of [21], we want to estimate $\mathbb{E}[G(X)] =$

[3] A related but slightly weaker definition of optimality is used in [25].

$\mathbb{E}[\exp(\log(G(X)))]$. Here we perform a measure change for the Brownian motion, so for ease of writing let us define $F(W) := \log(G(X(W)))$ with the natural convention that $F = -\infty$ when $G = 0$ and consider the more general problem of estimating,

$$\alpha(\epsilon) := \mathbb{E}[\exp(F(\sqrt{\epsilon}W)/\epsilon)], \qquad \text{for } \epsilon > 0.$$

When $\epsilon = 1$, this is our original problem, and the limit of this expression as $\varepsilon \to 0$ can be computed using Varadhan's lemma (this is referred to as small noise asymptotics). We now consider a general estimator for this quantity $\hat{\alpha}(\epsilon)$ (there is no requirement for $\hat{\alpha}$ to be based on a deterministic measure change). At this point we have no conditions on these estimators so we follow definition [21]*Definition 2.1.

***Definition* 4.3.** A family of estimators $\{\hat{\alpha}(\epsilon)\}$ is said to be *asymptotically relatively unbiased* if the following holds,

$$\frac{\mathbb{E}[\hat{\alpha}(\epsilon)] - \alpha(\epsilon)}{\alpha(\epsilon)} \to 0 \quad \text{as } \epsilon \to 0\,.$$

The above definition yields estimators that in some sense converge, but we are interested in comparing such estimators and for this we look at their second moment.

***Definition* 4.4.** A family of asymptotically relatively unbiased estimators $\{\hat{\alpha}_0(\epsilon)\}$ is said to be *asymptotically optimal* if,

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{E}[\hat{\alpha}_0(\epsilon)^2] = \inf_{\{\hat{\alpha}(\epsilon)\}} \limsup_{\epsilon\to 0} \epsilon \log \mathbb{E}[\hat{\alpha}(\epsilon)^2],$$

where the infimum is over all asymptotically relatively unbiased estimators.

One of the goals of this section will be obtaining conditions when measure changes of type (3.1) are asymptotically optimal. We shall use an argument similar to that given in [21]*pg 126. Consider an asymptotically unbiased estimator $\hat{\alpha}$, and define the difference $\Delta(\epsilon) := \mathbb{E}[\hat{\alpha}(\epsilon)] - \alpha(\epsilon)$. Jensen's inequality then yields a lower bound for the estimator:

$$\log(\mathbb{E}[\hat{\alpha}(\epsilon)^2]) \geq 2\log(\mathbb{E}[\hat{\alpha}(\epsilon)]) = 2\log\big(\alpha(\epsilon)\big) + O(\Delta(\epsilon)/\alpha(\epsilon)).$$

Thus,

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{E}[\hat{\alpha}(\epsilon)^2] \geq \limsup_{\epsilon\to 0} 2\epsilon \log(\alpha(\epsilon)).$$

Since the degenerate estimator $\hat{\alpha}(\epsilon) = \alpha(\epsilon)$ is asymptotically optimal, the definition of asymptotic optimality can be alternatively written as

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{E}[\hat{\alpha}_0(\epsilon)^2] = \limsup_{\epsilon\to 0} 2\epsilon \log(\alpha(\epsilon)).$$

The right-hand side of this expression, which corresponds to the small-noise limit of the original expectation $\mathbb{E}[G(X)]$ can be computed using Varadhan's lemma and Schilder's theorem: if $F$ is a continuous mapping satisfying the assumptions of Lemma 2.5 then

$$\limsup_{\epsilon\to 0} 2\epsilon \log(\alpha(\epsilon)) = \lim_{\epsilon\to 0} 2\epsilon \log \mathbb{E}\left[\exp\left(\frac{1}{\epsilon}F(\sqrt{\epsilon}W)\right)\right] = 2\sup_{u\in\mathbb{H}_T^d}\Big\{\log(G(X(u))) - \int_0^T |\dot{u}_t|^2 \mathrm{d}t\Big\}. \tag{4.1}$$

## 4.2. The decoupling algorithm

We first consider the decoupling algorithm presented in Section 3.1. We build $\mu_t^N$ from an independent $N$-particle system which is simulated under a numerical scheme (as argued in Section 3.1) and then consider the following approximation of SDE[4] (2.1),

$$\mathrm{d}\overline{X}_t = b(t, \overline{X}_t, \mu_t^N)\mathrm{d}t + \sigma \mathrm{d}W_t, \qquad X_0 = x_0\,. \tag{4.2}$$

In order to distinguish the current SDE from the previous particle approximation we introduce a so-called copy space (see for example [6]) $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$, where $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions and supporting a $N$-dimensional Brownian motion. The $N$-dimensional system of SDEs used to approximate the measure $\mu^N$ is then defined on this space, hence (4.2) is defined on the product space $(\Omega, \mathcal{F}, \mathbb{P}) \otimes (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

Our aim is now to minimize over $h \in \mathbb{H}_T^d$ the variance conditional on the trajectory of $\mu^N$:

$$\mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}\big[\,G(\overline{X})^2\mathcal{E}_T^{-1}\big|\tilde{\mathcal{F}}_T\big], \quad \mathrm{d}\mathcal{E}_t = \langle \dot{h}_t, \mathcal{E}_t \mathrm{d}W_t^{\mathbb{P}}\rangle, \quad \mathcal{E}_0 = 1,$$

and we make use of small noise asymptotics to define a tractable proxy for this variance, written as,

$$L(h;\mu^N) := \limsup_{\epsilon\to 0} \epsilon \log \mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}\Big[\exp\Big(\frac{1}{\epsilon}\Big(2\log(\overline{G}(\sqrt{\epsilon}W)) - \int_0^T \sqrt{\epsilon}\langle \dot{h}_t, \mathrm{d}W_t\rangle + \frac{1}{2}\int_0^T |\dot{h}_t|^2 \mathrm{d}t\Big)\Big)\Big|\tilde{\mathcal{F}}_T\Big], \tag{4.3}$$

where $\overline{G}(W) := G(\overline{X}(W))$. One should keep in mind that $\overline{G}$ also depends on $\mu^N$, however, we suppress this notation for ease of presentation.

***Remark* 4.1.** In (4.3), we have a conditional expectation, thus $L(h;\mu^N)$ is technically a random variable on $\tilde{\Omega}$. Because this random variable is independent of the Brownian motion and $\overline{G}$ is $\tilde{\mathbb{P}}$-a.s. continuous w.r.t. the Brownian motion (Section 6.2), we can apply Varadhan's lemma to the conditional probability and obtain a $\tilde{\mathbb{P}}$-almost sure result.

[4]The measure $\mu^N$ is a random measure but is independent of the process $\overline{X}$ thus we have decoupled the SDE with random coefficients.

**Theorem 4.5.** *Let Assumptions 4.1 and 4.2 hold. Assume that for all $m \geq 2$ we have $\sup_{0\leq t\leq T} \mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}[W^{(2)}(\mu_t^N, \delta_0)^m] < \infty$, that $(t,x) \mapsto b(t,x,\mu_t^N)$ is $\tilde{\mathbb{P}}$-a.s. jointly continuous and fix $\tilde{\omega} \in \tilde{\Omega}$ (and thus $\mu^N$). Furthermore assume that there exists $u \in \mathbb{H}_T^d$ such that $\overline{G}(u) > 0$. Then the following statements hold:*

*i. Let $h \in \mathbb{H}_T^d$ such that $\dot{h}$ is of finite variation. Then Varadhan's lemma holds for the small noise asymptotics, namely we can rewrite (4.3) as,*

$$L(h;\mu^N) = \sup_{u\in\mathbb{H}_T^d} \left\{ 2\log(\overline{G}(u)) + \frac{1}{2}\int_0^T |\dot{h}_t - \dot{u}_t|^2 \mathrm{d}t - \int_0^T |\dot{u}_t|^2 \mathrm{d}t \right\} \quad \tilde{\mathbb{P}}\text{-a.s.}. \tag{4.4}$$

*ii. There exists an $h^* \in \mathbb{H}_T^d$ which minimizes (4.4).*

*iii. Consider a simplified optimization problem*

$$\sup_{u\in\mathbb{H}_T^d} \left\{ 2\log(\overline{G}(u)) - \int_0^T |\dot{u}_t|^2 \mathrm{d}t \right\} . \tag{4.5}$$

*There exists a maximizer $h^{**}$ for this problem. If*

$$L(h^{**};\mu^N) = 2\log(\overline{G}(h^{**})) - \int_0^T |\dot{h}_t^{**}|^2 \mathrm{d}t\,, \tag{4.6}$$

*then $h^{**}$ defines an asymptotically optimal measure change and is the unique maximizer of (4.5).*

All of these results are $\tilde{\mathbb{P}}$-a.s. since the particle system yields a random measure from $\tilde{\Omega}$. The proof of this theorem requires several auxiliary results which we defer to Section 6.2. One should also note that the requirement for $\overline{G} > 0$ for some $u$ is not restrictive, it is purely there for technical reasons since one cannot have a maximiser if $\log(\overline{G}(u)) = -\infty$ for all $u \in \mathbb{H}_T^d$. The assumption that $\dot{h}$ has finite variation is necessary to establish the continuity of the functional in Varadhan's lemma.

***Remark* 4.2** (On $\mu^N$)**.** Condition $\sup_{0\leq t\leq T} \mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}[W^{(2)}(\mu_t^N, \delta_0)^m] < \infty$ for all $m \geq 2$ is not as restrictive as it may first appear. This condition holds under the one-sided Lipschitz assumption is if one generates the empirical measure using a so-called taming scheme (see [14]). It is then possible to show for any $m \geq 2$,

$$\sup_{0\leq t\leq T} \mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}[W^{(2)}(\mu_t^N, \delta_0)^m] \leq C^m m^m < \infty.$$

***Remark* 4.3** (Concavity of $\log(\overline{G})$ and asymptotic optimality)**.** Consider the problem of minimizing (4.4) and assume that one can interchange the inf and the sup. Then,

$$\begin{aligned}\inf_{h\in\mathbb{H}_T^d} L(h;\mu^N) &= \sup_{u\in\mathbb{H}_T^d}\inf_{h\in\mathbb{H}_T^d}\left\{2\log(\overline{G}(u))+\frac{1}{2}\int_0^T|\dot{h}_t-\dot{u}_t|^2\mathrm{d}t-\int_0^T|\dot{u}_t|^2\mathrm{d}t\right\}\\ &= \sup_{u\in\mathbb{H}_T^d}\left\{2\log(\overline{G}(u))-\int_0^T|\dot{u}_t|^2\mathrm{d}t\right\}\end{aligned}$$

because the inner problem is solved by choosing $h=u$. Therefore, a sufficient condition for an asymptotically optimal measure change of type (3.1) is the exchangeability of inf and sup above. Since $L$ is a convex function in $h$, and the integral terms in (4.4) are concave in $u$, a sufficient condition for such exchangeability is that $\log(\overline{G})$ is concave. Indeed, in the case of convex-concave functionals we can invoke the minimax principle to swap infimum and supremum, see [18]*pg. 175 for example.

In [25], the process $X$ was a geometric Brownian Motion and the authors were able to explicitly link the concavity of $\log(\overline{G})$ with the properties of the function $G$. Here the dependence of $\overline{G}$ on the Brownian motion is more complex, and it appears to be difficult to check concavity. Hence, in general one has to check numerically whether (4.6) holds. However, even if (4.6) fails, one can still use $h^{**}$ to construct a candidate importance sampling measure.

## 4.3. The complete measure change algorithm

Here we focus on the algorithm discussed in Section 3.2. Recall that we are interested in evaluating, $\mathbb{E}_\mathbb{P}[G(X)]$. We now change the measure to $\mathbb{Q}$ and calculate the variance,

$$\mathrm{Var}_\mathbb{Q}\Big[G(X)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\Big]=\mathbb{E}_\mathbb{P}\Big[G(X)^2\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\Big]-\mathbb{E}_\mathbb{P}\Big[G(X)\Big]^2.$$

Minimising the variance is equivalent to minimize the first term in the RHS. As a first step to constructing a tractable proxy for this variance we consider a particle approximation of $X$:

$$\begin{aligned}\mathrm{d}X_t^{i,N} &= b\Big(t,X_t^{i,N},\frac{1}{N}\sum_{j=1}^N\delta_{X_t^{j,N}}\Big)\mathrm{d}t+\sigma\mathrm{d}W_t^{i,\mathbb{P}}\,,\quad X_0^{i,N}=x_0\,, \qquad (4.7)\\ \mathrm{d}\mathcal{E}_t^i &= \langle\dot{h}_t,\mathcal{E}_t^i\mathrm{d}W_t^{i,\mathbb{P}}\rangle,\quad \mathcal{E}_0^i=1,\end{aligned}$$

where $W^{i,\mathbb{P}}\in\mathbb{W}_T^d$ denotes the driving $\mathbb{P}$-Brownian motion of particle $i$ with $i=1,\cdots,N$, and all $W^{i,\mathbb{P}}$s are independent of each other. We substitute $\mathbb{E}_\mathbb{P}[G^2(X)(\mathcal{E}_T)^{-1}]$ with the minimization of

$$\mathbb{E}_\mathbb{P}\big[\ G^2(X^{i,N})(\mathcal{E}_T^i)^{-1}\ \big]\,,\quad \text{over all } h\in\mathbb{H}_T^d. \qquad (4.8)$$

In order to use the LDP theory to minimize (4.8), for all $i \in \{1, \dots, N\}$, we define $\tilde{G}_i : (\mathbb{W}_T^d)^N \to \mathbb{R}$ by $\tilde{G}_i(W^1, \dots, W^N) := G(X^{i,N}(W^1, \dots, W^N))$. The small noise asymptotics for the functional (4.8) takes the following form for any $h \in \mathbb{H}_T^d$

$$\bar{L}(h) := \limsup_{\epsilon \to 0} \epsilon \log \left( \mathbb{E}_{\mathbb{P}} \left[ \exp \left( \frac{1}{\epsilon} \Big( 2 \log \big( \tilde{G}_i \big( \sqrt{\epsilon} W^1, \dots, \sqrt{\epsilon} W^N \big) \big) - \int_0^T \sqrt{\epsilon} \langle \dot{h}_t, \mathrm{d}W_t^i \rangle + \frac{1}{2} \int_0^T |\dot{h}_t|^2 \mathrm{d}t \Big) \right) \right] \right), \tag{4.9}$$

where we remark that the value of this expression does not depend on the choice of $i$. We then obtain the following result for $\bar{L}$ (compare with Theorem 4.5).

**Theorem 4.6.** *Fix $N \in \mathbb{N}$ and let Assumptions 4.1 and 4.2 hold. Assume that there exists $(u^1, \hat{u}) \in (\mathbb{H}_T^d)^2$ such that $\tilde{G}_1(u^1, \hat{u}, \dots, \hat{u}) > 0$. Then the following statements hold*

*i. Let $h \in \mathbb{H}_T^d$ such that $\dot{h}$ is of finite variation. Then Varadhan's lemma holds for the small noise asymptotics and we can rewrite (4.9) as*

$$\bar{L}(h) = \sup_{u \in (\mathbb{H}_T^d)^N} \Big\{ 2 \log(\tilde{G}_1(u^1, \dots, u^N)) + \frac{1}{2} \int_0^T |\dot{h}_t - \dot{u}_t^1|^2 \mathrm{d}t - \frac{1}{2} \int_0^T |\dot{u}_t^1|^2 \mathrm{d}t - \frac{1}{2} \int_0^T |\dot{u}_t|^2 \mathrm{d}t \Big\} , \tag{4.10}$$

*ii. There exists an $h^* \in \mathbb{H}_T^d$ which minimizes (4.10).*

*iii. Consider a simplified optimization problem*

$$\sup_{u^1 \in \mathbb{H}_T^d, \hat{u} \in \mathbb{H}_T^d} \left\{ 2 \log(\tilde{G}_1(u^1, \hat{u}, \dots, \hat{u})) - \int_0^T |\dot{u}_t^1|^2 \mathrm{d}t - \frac{N-1}{2} \int_0^T |\dot{\hat{u}}_t|^2 \mathrm{d}t \right\} . \tag{4.11}$$

*There exists a maximizer $(h^{**}, u^{**})$ for this problem. If*

$$\bar{L}(h^{**}) = 2 \log \big( \tilde{G}_1(h^{**}, u^{**}, \dots, u^{**}) \big) - \int_0^T |\dot{h}_t^{**}|^2 \mathrm{d}t - \frac{N-1}{2} \int_0^T |\dot{u}_t^{**}|^2 \mathrm{d}t \, . \tag{4.12}$$

*then $h^{**}$ is asymptotically optimal and is the unique maximizer of (4.11), where we have taken $i = 1$ without loss of generality.*

The proof of this theorem is deferred to Section 6.1. Similarly to the previous discussion if $\log(\tilde{G}_1)$ is a concave function in $u \in (\mathbb{H}_T^d)^N$, then we know that (4.12) holds (this is discussed at the end of Section 6.1). However, in general (4.12) is difficult to check since, even with $h^*$ fixed, $\bar{L}$ is still an $N$-dimensional optimisation problem, since (4.10) is supremum over $u \in (\mathbb{H}_T^d)^N$.

There is also a difficulty in quantifying how the measure change affects the propagation of chaos error, i.e., a measure change that is good for the statistical error may be damaging to the propagation of chaos error. We discuss this point further in Section 5.

## 4.4. Computing the optimal measure change

In this section we consider the special case when $G$ only depends on the terminal value of $X$: $G(X) = G(X_T)$. Our aim is to express the optimal measure change in a more explicit form, solving the optimization problems of Theorems 4.5 and 4.6 using the methods of deterministic optimal control (the reader can consult [20] or [33] for a background). More specifically, we use Pontryagin's maximum principle, which gives a set of differential equations that the optimal control must satisfy. Let us recall the main ideas following [33, p.102]. We start with the controlled dynamical system $x(t)$ which takes the following form:

$$\dot{x}(t) = b(t, x(t), u(t)), \qquad x(0) = x_0\,, \tag{4.13}$$

where $u$ is the control, defined in a metric space $(U, d)$. The aim of the controller is to minimize the *cost functional*

$$J\big(u(\cdot)\big) = \int_0^T f\big(t, x(t), u(t)\big)\mathrm{d}t + h\big(x(T)\big)\,, \tag{4.14}$$

$f$ is typically referred to as the *running cost* and $h$ the *terminal cost*. We make the following assumption.

***Assumption* 4.7.**

- $(U, d)$ is a separable metric space and $T > 0$.
- The maps $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are measurable and there exists a constant $L > 0$ and a modulus of continuity $\eta : [0, \infty) \to [0, \infty)$ such that,

$$|b(t, x, u) - b(t, \hat{x}, \hat{u})| + |f(t, x, u) - f(t, \hat{x}, \hat{u})| + |h(x) - h(\hat{x})| \le L|x - \hat{x}| + \eta(d(u, \hat{u}))$$
$$|b(t, 0, u)| + |f(t, 0, u)| \le L$$

  for all $t \in [0, T]$ $x, \hat{x} \in \mathbb{R}^n$, $u, \hat{u} \in U$.
- The maps $b$, $f$ and $h$ are $C^1$ in $x$ and there exists a modulus of continuity $\eta : [0, \infty) \to [0, \infty)$ such that,

$$\begin{aligned}|\partial_x b(t, x, u) - \partial_x b(t, \hat{x}, \hat{u})| &+ |\partial_x f(t, x, u) - \partial_x f(t, \hat{x}, \hat{u})| \\ &+ |\partial_x h(x) - \partial_x h(\hat{x})| \le \eta\big(|x - \hat{x}| + d(u, \hat{u})\big)\end{aligned}$$

  for all $t \in [0, T]$ $x, \hat{x} \in \mathbb{R}^n$, $u, \hat{u} \in U$.

As discussed in [33, p.102], Assumption 4.7 implies that (4.13) admits a unique solution and (4.14) is well defined. Let us introduce the set of admissible controls $\mathcal{U}[0, T] := \{u(\cdot) : [0, T] \to U \mid u \text{ is measurable}\}$. The optimal control problem is to find $u^* \in \mathcal{U}[0, T]$ that satisfies,

$$J(u^*) = \inf_{u \in \mathcal{U}[0,T]} J(u)\,. \tag{4.15}$$

Such $u^*$ is referred to as an *optimal control*, and the corresponding $x^*(\cdot) := x(\cdot\,; u^*)$ as an *optimal state trajectory*. We state the deterministic version of Pontryagin's maximum principle as in [33, p.103].

**Theorem 4.8.** *[Pontryagin's Maximum Principle] Let Assumption 4.7 hold and let $(x^*, u^*)$ be the optimal control-solution pair. Then, there exists a function $p : [0,T] \to \mathbb{R}^n$ satisfying the following,*

$$\begin{cases} \dot p(t) = -\partial_x b(t, x^*(t), u^*(t))^\intercal p(t) + \partial_x f(t, x^*(t), u^*(t)), & a.e.\ t \in [0,T] \\ p(T) = -\partial_x h(x^*(T))\,, \end{cases} \tag{4.16}$$

*and*

$$H(t, x^*(t), u^*(t), p(t)) = \max_{u\in U}\{H(t, x^*(t), u, p(t))\} \quad a.e.\ t \in [0,T]\,,$$

*where $H(t,x,u,p) := \langle p, b(t,x,u)\rangle - f(t,x,u)$ for any $(t,x,u,p) \in [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n$.*

Typically $p$ is referred to as the *adjoint function* and (4.16) the *adjoint equation*, and the function $H$ is called the *Hamiltonian*.

***Remark* 4.4** (An alternative approach)**.** The maximum principle is not the only way one can use to solve this problem. An alternative is by solving the so-called Hamilton-Jacobi-Bellman (HJB) equation. This approach is typically more difficult since the HJB is a *partial differential equation*.

**Maximum principle for Theorems 4.5 and 4.6.** The maximum principle allows to translate the simplified optimization problems of Theorems 4.5 and 4.6 into boundary value problems for ODE. One can observe that we are actually interested in $\dot u$ rather than $u$, that is, in the decoupled case we can write the controlled dynamics as

$$X_t(\dot u) = x_0 + \int_0^t b(s, X_s(\dot u), \mu^N_s)\mathrm{d}s + \int_0^t \sigma \dot u_s \mathrm{d}s\,.$$

The theory above is for infimum while we are interested in supremum, therefore we use the fact that $\sup\{f\} = -\inf\{-f\}$.

▷ *For the decoupling algorithm* Theorem 4.8 yields the following equations for the optimization problem (4.5), which hold $\tilde{\mathbb{P}}$-a.s., for the adjoint function and trajectory under optimal control $\dot u^*$ (for a given $\mu^N$),

$$\text{(Decoupled)} \quad \begin{cases} \dot p_t = -\partial_x b(t, X_t(\dot u^*), \mu^N_t)^\intercal p_t\,, & p_T = \dfrac{2}{G(X_T(\dot u^*))}\partial_x G(X_T(\dot u^*))\,, \\ \dot X_t = b(t, X_t, \mu^N_t) + \frac{1}{2}\sigma \begin{bmatrix} \langle p_t, \sigma e_1\rangle \\ \vdots \\ \langle p_t, \sigma e_d\rangle \end{bmatrix}, & X_0 = x_0\,, \end{cases} \tag{4.17}$$

where $e_i$ denotes the $d$-dimensional vector with 1 at position $i$ and zero elsewhere. One can observe, the optimal control is related to $p$ through, $(\dot{u}^*_t)_i = \frac{1}{2}\langle p_t, \sigma e_i\rangle$ for $i \in \{1, \dots, d\}$.

▷ *For the complete measure change algorithm* the argument is similar to the above one, but here we also need to deal with the measure term; additionally in this case the system is deterministic as opposed to above where it holds $\tilde{\mathbb{P}}$-a.s. Noting that we have two controls to optimise over (recall Theorem 4.6) we obtain more complex expressions. Theorem 4.8 yields the following system of ODEs for the optimization problem (4.11).

$$\begin{cases} \dot{p}^1_t = -\partial_{X^1} b(t, X^1_t, \frac{1}{N}\delta_{X^1_t} + \frac{N-1}{N}\delta_{\hat{X}_t})p^1_t - \partial_{X^1} b(t, \hat{X}_t, \frac{1}{N}\delta_{X^1_t} + \frac{N-1}{N}\delta_{\hat{X}_t})p^2_t\,, & p^1_T = \frac{2\partial_x G(X^1_T)}{G(X^1_T)}\,, \\ \dot{p}^2_t = -\partial_{\hat{X}} b(t, X^1_t, \frac{1}{N}\delta_{X^1_t} + \frac{N-1}{N}\delta_{\hat{X}_t})p^1_t - \partial_{\hat{X}} b(t, \hat{X}_t, \frac{1}{N}\delta_{X^1_t} + \frac{N-1}{N}\delta_{\hat{X}_t})p^2_t\,, & p^2_T = 0\,, \\ \dot{X}^1_t = b(t, X^1_t, \frac{1}{N}\delta_{X^1_t} + \frac{N-1}{N}\delta_{\hat{X}_t}) + \frac{1}{2}\sigma \begin{bmatrix} \langle p^1_t, \sigma e_1\rangle \\ \vdots \\ \langle p^1_t, \sigma e_d\rangle \end{bmatrix}, & X^1_0 = x_0\,, \\ \dot{\hat{X}}_t = b(t, \hat{X}_t, \frac{1}{N}\delta_{X^1_t} + \frac{N-1}{N}\delta_{\hat{X}_t}) + \frac{1}{N-1}\sigma \begin{bmatrix} \langle p^2_t, \sigma e_1\rangle \\ \vdots \\ \langle p^2_t, \sigma e_d\rangle \end{bmatrix}, & \hat{X}_0 = x_0\,, \end{cases} \tag{4.18}$$

similarly we obtain, $(\dot{u}^*_t)_i = \frac{1}{2}\langle p^1_t, \sigma e_i\rangle$ and $(\dot{\hat{u}}^*_t)_i = \frac{1}{N-1}\langle p^2_t, \sigma e_i\rangle$ as the optimal controls for $i \in \{1, \dots, d\}$. From Theorem 4.6 we obtain the measure change as $\dot{h} = \dot{u}$.

The difference between (4.17) and (4.18) comes from the fact that for the complete measure change we have a higher dimensional problem with two controls and two SDEs. Recall, when one wishes to assess asymptotic optimality, (4.10) is still an $N$-dimensional problem.

***Remark* 4.5** (Accuracy of Change of Measure)**.** In [25], the authors were able to obtain explicit solutions in certain situations, but here, due to the increased complexity, we expect this to rarely be the case. This forces us to invoke a numerical method and hence asymptotic optimality can never be guaranteed, however, provided we are close (same error order as the numerical scheme) one can be reasonably confident that the measure change is close to optimal.

## 5. Example: Kuramoto model

The Kuramoto model is a special case of a so-called system of coupled oscillators. Such models are of particular interest in physics and are used to study many different phenomena such as active rotator systems, charge density waves and complex biological systems amongst other things, see [28] for further details. The SDE corresponding to the Kuramoto model is

$$\mathrm{d}X_t = \left(K\int_{\mathbb{R}} \sin(y - X_t)\mu^X_{t,\mathbb{P}}(\mathrm{d}y) - \sin(X_t)\right)\mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{P}}_t\,, \quad t \in [0,T], \quad X_0 = x_0\,,$$

where $K$ is the coupling strength and $\sigma$ has the physical interpretation of the temperature in the system. We consider a terminal condition $G(x) = a\exp(bx)$. Our goal is to obtain an asymptotically optimal change of measure, which improves the estimation of $\mathbb{E}_{\mathbb{P}}[G(X_T)]$.

This model clearly satisfies Assumptions 4.1 as long as $\sigma$ is nondegenerate with non-negative entries. From a strict point of view, $G$ does not satisfy Assumption 4.2 which is for polynomial growth only (for any power). We apply our methodology nonetheless, otherwise one could replace $G$ by a high power polynomial.

Let us now apply the theory from the previous section to calculate the optimal change of measure. We should point out here that we do not have the concavity required for asymptotic optimality to hold automatically, therefore we need to check this condition numerically.

By our previous discussion, to apply the decoupling algorithm here we generate a set of $N$ weakly interacting SDEs which we denote by $Y^{i,N}$ and approximate the original SDE by,

$$\mathrm{d}\bar{X}_t = \left(\frac{K}{N}\sum_{i=1}^{N}\sin(Y_t^{i,N} - \bar{X}_t) - \sin(\bar{X}_t)\right)\mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{P}}\,, \quad t\in[0,T], \quad \bar{X}_0 = x_0\,.$$

Let us now apply the theory from the previous section to calculate the optimal change of measure. Our optimal control argument implies solving $\tilde{\mathbb{P}}$-a.s.

$$\text{(Decoupled)} \quad \begin{cases} \dot{p}_t = \left(\frac{K}{N}\sum_{i=1}^{N}\cos(Y_t^{i,N} - X_t) + \cos(X_t)\right)p_t\,, & p_T = 2b\,,\\ \dot{X}_t = \left(\frac{K}{N}\sum_{i=1}^{N}\sin(Y_t^{i,N} - X_t) - \sin(X_t)\right) + \frac{1}{2}\sigma^2 p_t\,, & X_0 = x_0\,. \end{cases}$$

The complete measure change algorithm yields the following system, (Complete)

$$\begin{cases} \dot{p}_t^1 = K\big(\frac{N-1}{N}\cos(\hat{X}_t - X_t^1) + \cos(X_t^1)\big)p_t^1 - \frac{K}{N}\cos(X_t^1 - \hat{X}_t)p_t^2\,, & p_T^1 = 2b\,,\\ \dot{p}_t^2 = -K\frac{N-1}{N}\cos(\hat{X}_t - X_t^1)p_t^1 + K\big(\frac{1}{N}\cos(X_t^1 - \hat{X}_t) + \cos(\hat{X}_t)\big)p_t^2\,, & p_T^2 = 0\,,\\ \dot{X}_t^1 = K\big(\frac{N-1}{N}\sin(\hat{X}_t - X_t^1) - \sin(X_t^1)\big) + \frac{1}{2}\sigma^2 p_t^1\,, & X_0^1 = x_0\,,\\ \dot{\hat{X}}_t = K\big(\frac{1}{N}\sin(X_t^1 - \hat{X}_t) - \sin(\hat{X}_t)\big) + \frac{1}{N-1}\sigma^2 p_t^2\,, & \hat{X}_0 = x_0\,, \end{cases}$$

To show the numerical advantages one can achieve by using importance sampling we consider how the time taken and the estimate given by the algorithms change with the number of particles $N$.

For this example we use, $T = 1$, $\bar{X}_0 = 0$, $K = 1$, $\sigma = 0.3$, $a = 0.5$ and $b = 10$. For the numerics we use an Euler scheme with step size of $\Delta t = 0.02$. The systems of equations are solved using MATLAB's `bvp4c` function, which uses a Lobatto IIIA method, see [30] for example. For the importance sampling, we use the particle positions from the first Monte Carlo simulation as the empirical law.

We recall that the decoupling importance sampling requires two runs, here we use the same $N$ for both of these. The first note one can make is how the time scales when

| | Monte Carlo | | | Decoupled | | | Complete | | |
|---|---|---|---|---|---|---|---|---|---|
| N | Payoff | Error | Time | Payoff | Error | Time | Payoff | Error | Time |
| $1 \times 10^3$ | 1.5066 | 0.1490 | 3 | 1.5729 | 0.0028 | 9 | 1.5419 | 0.0024 | 3 |
| $5 \times 10^3$ | 1.5895 | 0.0626 | 27 | 1.5840 | 0.0013 | 54 | 1.5710 | 0.0013 | 28 |
| $1 \times 10^4$ | 1.6813 | 0.0693 | 76 | 1.5728 | 0.0009 | 153 | 1.5860 | 0.0009 | 75 |
| $5 \times 10^4$ | 1.5899 | 0.0200 | 1 025 | 1.5820 | 0.0004 | 2 052 | 1.5738 | 0.0004 | 1 062 |
| $1 \times 10^5$ | 1.5807 | 0.0176 | 3 433 | 1.5731 | 0.0003 | 6 935 | 1.5882 | 0.0003 | 3 644 |

**Table 1.** Results from standard Monte Carlo and the importance sampling algorithms. Time is measured in seconds and error refers to square root of the variance.

increasing the number of particles, namely one can truly observe the $N^2$ complexity[5]. As expected the decoupling algorithm takes approximately twice as long as the standard Monte Carlo (computing the change of measure is not time consuming). Following this point we also observe that the complete measure change has roughly the same computational complexity as standard Monte Carlo. The other key point is the reduction in variance (standard error) one obtains with importance sampling. For this example we see that both importance sampling schemes reduce the variance by several orders of magnitude. Further, the decoupling algorithm's efficiency in terms of computational costs can be improved. The second run of simulations can be carried out with less samples than the first run; we do not explore optimality in this regard.

Finally, we checked the asymptotic optimality (for the decoupling) numerically and there is only a difference of $O(10^{-4})$ between the two sides in (4.12). As this is the same accuracy as the numerical solver we have used, we believe this solution is close to the optimal one.

$\triangleright$ *Estimating the propagation of chaos error.* As was mentioned in the introduction, theoretically the statistical error and the propagation of chaos error converge to zero at the same rate. We now use this example to show that the statistical error dominates. Since the Euler scheme is the same in all examples we can neglect the bias caused by that. We can then decompose the error as

$$\frac{1}{N}\sum_{i=1}^{N} G(\bar{X}^{i,N}) - \mathbb{E}_{\mathbb{P}}[G(\bar{X}^1)]$$
$$= \frac{1}{N}\sum_{i=1}^{N} G(\bar{X}^{i,N}) - \mathbb{E}_{\mathbb{P}}[G(\bar{X}^{1,N})] + \mathbb{E}_{\mathbb{P}}[G(\bar{X}^{1,N})] - \mathbb{E}_{\mathbb{P}}[G(\bar{X}^1)]\,.$$

The first difference on the RHS is the statistical error, and the second one is the propagation of chaos error. It is then clear that if one considers $M$ realisations of $\frac{1}{N}\sum_{i=1}^{N} G(\bar{X}^{i,N})$ and takes the average, for $M \to \infty$ the resulting estimator converges to $\mathbb{E}_{\mathbb{P}}[G(\bar{X}^{1,N})]$ and the error reduces to the propagation of chaos error. To show the propagation of chaos

[5] For the particular case of the Kuramoto model, the trigonometric identity $\sin(y-x) = \sin(y)\cos(x) - \sin(x)\cos(y)$ allows to decouple the measure dependency from the solution process and to simulate the equation with a complexity of $1/N$ rather than the $1/N^2$ we have; this has been exploited in [2]. We chose to present our results without this trick to highlight the generality in which our method applies.

error is negligible compared to the statistical error here, we repeat the simulation for $N = 5 \times 10^3$ particles, $M = 10^3$ times and we obtain an average terminal value of 1.5772 (with an average standard error of 0.06533, which agrees with the result in Table 1). Comparing this to the $10^5$ decoupled entry (which has very low statistical error) in Table 1, we can conclude the propagation of chaos error at least an order of magnitude smaller than the statistical error.

*Another example: a terminal condition function with steep slope*

Let us consider the terminal condition $G(x) = \big(\tanh(a(x-b)) + 1\big)/2$, for $a$ large ($G$ can be understood as a mollified indicator function). Then $\mathbb{E}_{\mathbb{P}}[G(X_T)] \approx \mathbb{P}(X_T \geq b)$. We take the same set up as before but with $a = 15$ and $b = 1$ and note that the terminal condition for the adjoint process takes the form,

$$p_T = 2a\Big(1 - \tanh\big(a\big(X_T(\dot{u}^*) - b\big)\big)\Big)\,.$$

We obtain the following table (we omit the times here since they are similar).

| | Monte Carlo ($\times 10^{-9}$) | | Decoupled ($\times 10^{-9}$) | | Complete ($\times 10^{-9}$) | |
|---|---|---|---|---|---|---|
| N | Payoff | Error | Payoff | Error | Payoff | Error |
| $1 \times 10^3$ | 1.015 | 0.671 | 3.864 | 0.0250 | 8.456 | 0.101 |
| $5 \times 10^3$ | 1.093 | 0.752 | 3.952 | 0.0112 | 5.564 | 0.0185 |
| $1 \times 10^4$ | 8.829 | 7.071 | 3.910 | 0.0077 | 32.956 | 0.1520 |
| $5 \times 10^4$ | 1.106 | 0.271 | 3.970 | 0.0035 | 2.101 | 0.0024 |
| $1 \times 10^5$ | 5.158 | 1.990 | 3.901 | 0.0024 | 16.781 | 0.019 |

**Table 2.** Results from standard Monte Carlo and the importance sampling algorithms. Note that for ease of presentation the payoff and error are all scaled by the factor $10^{-9}$.

The results in Table 2 highlight the key differences in the algorithms. Clearly this is a difficult problem for standard Monte Carlo. The reason is that although $G$ is mollified it still changes value quickly over a small interval. For example $G(0.25) \approx 10^{-10}$, but $G(0.5) \approx 10^{-7}$ and $G(0.75) \approx 10^{-4}$, hence a reasonably small change in the value of the SDE can influence the outcome significantly. Besides, the probability of the solution attaining a region with high value of $G$ is quite low: for the standard Monte Carlo run, only 60 of the $100{,}000$ particles were above $1/2$ at the terminal time and none were above $3/4$. Hence standard Monte Carlo does not provide much information about the most important region of the function, and importance sampling is likely to lead to a considerable improvement.

The importance sampling schemes indeed provide reduced statistical errors, however, this example highlights the differences between them. Although the complete measure change does have a smaller statistical error than standard Monte Carlo, the overall error is very high, as seen from wildly oscillating values of the estimator. We conjecture that in this setting the propagation of chaos error dominates the statistical error, because the importance sampling measure is very far from the original measure, which deteriorates the quality of the estimation of the coefficients of the MV-SDE. Here, the decoupled algorithm appears to be superior since the estimated values are consistent and the error decreases in the expected manner.

***Remark* 5.1** (Requirement for improved simulation)**.** It is clear from these examples that combining importance sampling with MV-SDEs can provide a major reduction in the required computational cost. When using decoupling, unfortunately one has to approximate the law first, which is computationally expensive. Hence, one may look towards more sophisticated simulation techniques to speed up the first run, for example the methods of [24] or towards multilevel Monte Carlo [26]. However, with the ability to almost eliminate the variance one should always keep in mind the benefits from importance sampling.

# 6. Proof of Main Results

We now provide the proofs of our two main theorems. Throughout we work under the $\mathbb{P}$-measure and we omit it as a superscript in our Brownian motions. Some arguments align with those of [25] and we quote them where appropriate.

## 6.1. Complete measure change algorithm - proofs for Theorem 4.6

Continuity of the SDE w.r.t. Brownian motion is key as it allows to apply directly the contraction principle transferring Schilder's LDP for the Brownian motion to an LDP for the solution of the SDE; otherwise difficulties would arise when using Varadhan's lemma. All results holds under Assumption 4.1.

***Lemma* 6.1.** Fix $N \in \mathbb{N}$, let Assumption 4.1 hold and let $X \in \mathbb{S}^p$ for $p \geq 2$ denote the $N$-dimensional strong solution to the SDE system defined in (4.7).

Then $X$ is continuous w.r.t. the set of $N$ Brownian motions in the uniform topology.

***Proof.*** We wish to prove that the strong solution to (4.7) are continuous images of the trajectories of the driving Brownian motions. In fact, using a know trick from random dynamical systems, we can construct them pathwise for each $\omega \in \Omega$. Defining $Y^i := X^i - \sigma W^{i,\mathbb{P}}$, then $Y^i$ has dynamics

$$\begin{aligned} Y_t^{i,N}(\omega) = X_t^i(\omega) - \sigma W_t^{i,\mathbb{P}}(\omega) &= x_0 + \int_0^t b\left(s, X_s^{i,N}(\omega), \frac{1}{N}\sum_{j=1}^N \delta_{X_s^{j,N}(\omega)}\right) \mathrm{d}s \\ &= x_0 + \int_0^t b\left(s, Y_s^{i,N} + \sigma W_s^{i,\mathbb{P}}, \frac{1}{N}\sum_{j=1}^N \delta_{(Y_s^{i,N}+\sigma W_s^{i,\mathbb{P}})}\right) \mathrm{d}s, \end{aligned}$$

which is an ODE with random coefficients. The proof follows by appealing to Lemma 6.2 below in combination with assumption on the non-degeneracy of $\sigma$. □

***Lemma* 6.2.** Let Assumption 4.1 hold and take $g = (g^1, \cdots, g^N) \in C_0([0,T], (\mathbb{R}^d)^N)$, $x_0 \in \mathbb{R}^d$. Then there exists a unique $f = (f^1, \cdots, f^N) \in C([0,T], (\mathbb{R}^d)^N)$ which satisfies for $t \in [0,T]$

$$f^i(t) = x_0 + \int_0^t b\Big(s, f^i(s), \frac{1}{N}\sum_{j=1}^N \delta_{f^j(s)}\Big)\mathrm{d}s + g^i(t), \qquad i = 1, \cdots, N.$$

The mapping

$$F : C_0\big([0,T], (\mathbb{R}^d)^N\big) \to C\big([0,T], (\mathbb{R}^d)^N\big), \quad g \mapsto f$$

is continuous and one-to-one. Moreover, the map $F$ is locally Lipschitz.

***Proof.*** Let $t \in [0,T]$ and $i = 1, \cdots, N$. The proof takes several steps.

*Step 1: Short time existence.* From the continuity of the involved maps, Carathéodory's existence result for ODEs yields the existence of a solution $f$ given $g$ over a small time interval.

*Step 2: Uniqueness* follows by a direct application of the one-sided Lipschitz condition. Namely, given $f, \hat{f}$ two solutions to the ODE for a fixed $g$, then computing the derivative of $|f(t) - \hat{f}(t)|^2$, using the one-sided Lipschitz condition (where one adds and subtracts $g^i$ in the inner product) and Grönwall's inequality yields that $|f(t) - \hat{f}(t)|^2 = 0$ and hence uniqueness. Note that each $i$-th system depends only on $f^i, g^i$ and a cross dependency on the remaining terms via the average component.

*Step 3: A priori estimate and existence over longer intervals.* Define the absolutely continuous (wrt the Lebesgue measure) map $\psi := f - g$, then we have that $\psi$ satisfies the ordinary differential equation, (recall that $g(0) = 0$)

$$\psi^i(t) = x_0 + \int_0^t b\Big(s, \psi^i(s) + g^i(s), \nu^{\psi+g}(s)\Big)\mathrm{d}s \ \text{ with } \ \nu^{\psi+g}(s) := \frac{1}{N}\sum_{j=1}^N \delta_{\psi^j(s)+g^j(s)}, \tag{6.1}$$

where an estimate for $\|\psi\|_\infty$ yields easily an inequality for $\|f\|_\infty$ via the triangle inequality. Given $g \in C_0$, we compute $|\psi^i(t)|^2$,

$$\begin{aligned}
|\psi^i(t)|^2 &= |x_0|^2 + \int_0^t 2\Big\langle \psi^i(s), b\big(s, \psi^i(s) + g^i(s), \nu^{\psi+g}(s)\big)\Big\rangle \mathrm{d}s \\
&\le |x_0|^2 + \int_0^t 2L|\psi^i(s) + g^i(s)|^2 + |\psi^i(s)|^2 + |b\big(s, g^i(s), \nu^{\psi+g}(s)\big)|^2 \mathrm{d}s \qquad (6.2)\\
&\le |x_0|^2 + \int_0^t (1+4L)|\psi^i(s)|^2 + C(1 + |g^i(s)|^{2(q+1)}) + 2L\frac{1}{N}\sum_{j=1}^N |\psi^j(s) + g^j(s)|^2 \mathrm{d}s, \qquad (6.3)
\end{aligned}$$

where as before, on the first line, we added $\pm g$ to the LHS of the $\langle , \rangle$-bracket and $\pm b\big(\cdot, g(\cdot), \frac{1}{N}\sum_{j=1}^N \delta_{\psi^j(\cdot)+g^j(\cdot)}\big)$ to its RHS, then to get to the second line we dominated using the one-sided Lipschitz condition and Cauchy-Schwarz. In the last line we used the growth conditions of $b$, that continuous maps in compact intervals are bounded and the properties of the Wasserstein metric.

The average term appearing in the last term creates an additional problem solved by first averaging the ODEs over $i$, obtaining an estimate for the term corresponding to the average, then returning to the initial equation and inject it to obtain the sought estimate for $\|\psi^i\|_\infty$. Hence, averaging over $i$ estimate, setting $\bar{\psi}^N(t) := \frac{1}{N}\sum_{i=1}^d |\psi^i(t)|^2$, using that $|a+b| \leq 2a^2 + 2b^2$, re-arranging and Grönwall we have

$$\bar{\psi}^N(t) \leq C\big(1 + |x_0|^2 + \int_0^t \frac{1}{N}\sum_{i=1}^N |g^i(s)|^{2q+2}\mathrm{d}s\big) + C\int_0^t \bar{\psi}^N(s)\mathrm{d}s \tag{6.4}$$

$$\Rightarrow \quad \|\bar{\psi}^N\|_\infty = \sup_{0\leq t\leq T} |\psi^N(t)| \leq C\big(1 + |x_0|^2 + T\sup_{0\leq s\leq T}\frac{1}{N}\sum_{i=1}^N |g^i(s)|^{2q+2}\big)e^{CT}.$$

Injecting the estimate for $\|\bar{\psi}^N\|_\infty$ in (6.3), dominating and re-arranging we have

$$|\psi^i(t)|^2 \leq C\Big(1 + |x_0|^2 + \|\bar{\psi}^N\|_\infty + \int_0^t |g^i(s)|^{2q+2}\mathrm{d}s\Big) + C\int_0^t |\psi^i(s)|^2\mathrm{d}s. \tag{6.5}$$

We conclude via Grönwall that $\|\psi^i\|_\infty \leq C_R$ for some $C_R > 0$ where all $g^i$ belong to some Ball (in the uniform topology) of radius $R$ centered around zero.

It is now clear that one can extend the solution of Step 1 from the small interval to any interval of general length. One can simply repeat the above procedure a finite number of times on finitely many adjacent intervals with recursively chosen boundary conditions.

*Step 4. Continuity of $\psi$ and $f$ in the uniform topology.* We continue using the construction provided by (6.1) where the continuity of $\psi$ translates immediately to continuity of $f$ via linearity. Let $g, \hat{g} \in C_0$ sit in some ball of radius $R' > 0$ (centered around $g$) and associate to them $\psi, \hat{\psi}$ the respective solutions to (6.1) and respective empirical measures $\nu^{\psi+g}, \nu^{\hat{\psi}+\hat{g}}$. Define $\delta g := g - \hat{g}$ and $\delta\psi := \psi - \hat{\psi}$. As in the previous step, we compute $|\psi^i(t) - \hat{\psi}^i(t)|^2$,

$$\begin{aligned}
|\delta\psi^i(t)|^2 &= 0 + \int_0^t \Big\langle \delta\psi^i(s), b\Big(s, \psi^i(s) + g^i(s), \nu^{\psi+g}(s)\Big) - b\Big(s, \hat{\psi}^i(s) + \hat{g}^i(s), \nu^{\hat{\psi}+\hat{g}}(s)\Big)\Big\rangle \mathrm{d}s \\
&\leq C\int_0^t |\delta\psi^i(s)|^2 + |\delta g^i(s)|^2 \\
&\qquad + \Big\langle \delta\psi^i(s), b\Big(s, \hat{\psi}^i(s) + \hat{g}^i(s), \nu^{\psi+g}(s)\Big) - b\Big(s, \hat{\psi}^i(s) + \hat{g}^i(s), \nu^{\hat{\psi}+\hat{g}}(s)\Big)\Big\rangle \mathrm{d}s \\
&\qquad - \Big\langle \delta g^i(s), b\Big(s, \psi^i(s) + g^i(s), \nu^{\psi+g}(s)\Big) - b\Big(s, \hat{\psi}^i(s) + \hat{g}^i(s), \nu^{\psi+g}(s)\Big)\Big\rangle \mathrm{d}s \\
&\leq C\int_0^t |\delta\psi^i(s)|^2 + |\delta g^i(s)|^2 + |\delta\psi^i(s)|^2 + |W^{(2)}(\nu^{\psi+g}(s), \nu^{\hat{\psi}+\hat{g}}(s))|^2 + |\delta g^i(s)|B(s)\mathrm{d}s,
\end{aligned}$$

where to the first line we added $\pm\delta g$ to the LHS of the $\langle,\rangle$-bracket, $\pm b(s, \hat{\psi}^i(s) + \hat{g}^i(s), \nu^{\psi+g}(s))$ to its RHS and used the one-sided Lipschitz condition; on the final line we used that $\mu \mapsto b(\cdot,\cdot,\mu)$ is $W^{(2)}$-Lipschitz and $B(s) := C\big(1+|\psi^i(s)+g^i(s)|^q+|\hat{\psi}^i(s)+\hat{g}^i(s)|^q\big)\big(|\delta\psi^i(s)|+|\delta g^i(s)|\big)$ from the Locally Lipschitz condition. In fact, given the estimates for $\|\psi\|_\infty, \|\hat{\psi}\|_\infty$ obtained in Step 3, we have $B(s) \le C_{R,R'}\big(|\delta\psi^i(s)|+|\delta g^i(s)|\big)$ ($R, R'$ are the Balls mentioned above). Hence,

$$|\delta g^i(s)|B(s) \le C_{R,R'}|\delta g^i(s)|\big(|\delta\psi^i(s)|+|\delta g^i(s)|\big) \le C_{R,R'}|\delta g^i(s)|^2 + C_{R,R'}|\delta\psi^i(s)|^2.$$

Using the properties of the Wasserstein metric we easily find that there exists $C>0$ for any $s\in[0,T]$

$$|W^{(2)}(\nu^{\psi+g}(s),\nu^{\hat{\psi}+\hat{g}}(s))|^2 \le C\|\delta g\|_\infty^2 + C\frac{1}{N}\sum_{j=1}^N |\delta\psi^j(s)|^2.$$

Joining all the estimates and re-organizing we have

$$|\delta\psi^i(t)|^2 \le C_{R,R'}\Big(\|\delta g\|_\infty^2 + \int_0^t |\delta\psi^i(s)|^2 + \frac{1}{N}\sum_{j=1}^N |\delta\psi^j(s)|^2 \mathrm{d}s\Big).$$

The proof now follows as in Step 3. We average the inequality over $i$ and use Grönwall to obtain

$$\Big\|\frac{1}{N}\sum_{j=1}^N |\delta\psi^j|^2\Big\|_\infty \le C_{R,R'}e^T\|\delta g\|_\infty^2,$$

which is then injected in the above one yielding, after Grönwall, for yet another constant $C_{R,R'}>0$ uniformly in time

$$\|\psi^i-\hat{\psi}^i\|_\infty^2 \le C_{R,R'}\|g-\hat{g}\|_\infty^2.$$

We can now conclude the aforementioned continuity of $F$. One argues that for any radius $R'>0$ and for any sequence $(g^n)_{n\ge1}$ converging to $g$ in the uniform topology, there exists $M$ large enough such that all elements $(g^n)_{n>M}$ are contained in the ball centered around $g$ and radius $R'$, then one can apply the above inequality and conclude. Moreover, it is also clear that $g\mapsto F(g)=f$ is locally continuous uniformly on compacts of the uniform topology. □

We state an auxiliary bounding lemma, specialized for Cameron-Martin maps, which in essence is a by-product of the previous proof.

***Lemma* 6.3.** Fix $N\in\mathbb{N}$ and $u\in\big(\mathbb{H}_T^d\big)^N$, let $b$ and $\sigma$ satisfy Assumption 4.1 and consider the following system,

$$X_t^{i,N}(u_t) = x_0 + \int_0^t b\Big(s, X_s^{i,N}(u_s), \frac{1}{N}\sum_{j=1}^N \delta_{X_s^{j,N}(u_s)}\Big)\mathrm{d}s + \sigma u_t^i \quad \text{for } i=1,\dots,N,$$

with some abuse of notation and we have used $X_t^{i,N}(u_t) := X_t^{i,N}((u_s)_{s\le t})$. Then the following bound holds,

$$\sup_{1\le i\le N} |X_t^{i,N}(u_t)|^2 \le C\big(1+\int_0^t \sup_{1\le i\le N} |\sigma u_s^i|^{2q+2}\mathrm{d}s\big)e^C + C \sup_{1\le i\le N} |\sigma u_t^i|^2,$$

where the involved constants $C$ are uniform in time.

***Proof.*** Let $t\in[0,T]$ and $i=1,\dots,N$. We have $|X_t^{i,N}(u_t)| \le 2|X_t^{i,N}(u_t)-\sigma u_t^i|^2+2|\sigma u_t^i|^2$ and in relation to the proof of Lemma 6.2 one needs to take $g^i=\sigma u_t^i$ and revisit step 3: start at (6.4) and apply Grönwall directly to it, inject the estimate in (6.3) to get a slight variant of (6.5), we have then

$$\begin{aligned}\sup_{1\le i\le N} |X_t^{i,N}(u_t)| &\le \sup_{1\le i\le N}\Big\{2|X_t^{i,N}(u_t)-\sigma u_t^i|^2+2|\sigma u_t^i|^2\Big\}\\ &\le C\big(1+\int_0^t \sup_{1\le i\le N} |\sigma u_s^i|^{2q+2}\mathrm{d}s\big)e^C + C \sup_{1\le i\le N} |\sigma u_t^i|^2,\end{aligned}$$

where the involved constants $C$ are uniform in the time variable. □

We show that how one can bound an arbitrary moment of the particle system.

***Lemma* 6.4.** Fix $N\in\mathbb{N}$ and let Assumption 4.1 hold. Then for any $m\ge 2$ we obtain the following bound for any $t\in[0,T]$,

$$\mathbb{E}_{\mathbb{P}}[|X_t^{1,N}(\sqrt{\epsilon}W^1,\dots,\sqrt{\epsilon}W^N)|^m] \le C(C^m+m+\epsilon m^2)\exp\big(C(m+\epsilon m^2)\big)\,.$$

***Proof.*** For ease of notation we will write $X_t^{1,N} := X_t^{1,N}(\sqrt{\epsilon}W^1,\dots,\sqrt{\epsilon}W^N)$. By Itô's lemma we obtain,

$$\begin{aligned}|X_t^{1,N}|^m =&|x_0|^m + \int_0^t m|X_s^{1,N}|^{m-2}\big\langle X_s^{1,N}, b(s,X_s^{1,N},\frac{1}{N}\sum_{j=1}^N \delta_{X_s^{j,N}})\big\rangle \mathrm{d}s\\ &+m\int_0^t |X_s^{1,N}|^{m-2}\big\langle X_s^{1,N},\sigma\sqrt{\epsilon}\mathrm{d}W_s^1\big\rangle\\ &+\frac{1}{2}\int_0^t m|X_s^{1,N}|^{m-4}\big((m-2)\big(|\sigma|^2|X_s^{1,N}|^2-\langle X_s^{1,N},\sigma\sigma^\intercal X_s^{1,N}\rangle\big)+(m-1)|\sigma|^2|X_s^{1,N}|^2\big)\epsilon\mathrm{d}s.\end{aligned}$$

By adding and subtracting drift terms (similar to the proof of Proposition 3.1), Cauchy-Schwarz inequality and taking expectations we obtain,

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}[|X_t^{1,N}|^m]\\ &\le |x_0|^m + C\int_0^t m\mathbb{E}_{\mathbb{P}}\big[|X_s^{1,N}|^m + |X_s^{1,N}|^{m-1}\Big(\frac{1}{N}\sum_{j=1}^N |X_s^{j,N}|^2\Big)^{\frac{1}{2}}+1\big] + \epsilon m^2\mathbb{E}_{\mathbb{P}}\big[|X_s^{1,N}|^{m-2}\big]\mathrm{d}s,\end{aligned}$$

where we have used that $\int_0^t \mathbb{E}_\mathbb{P}[m^2\epsilon|\sigma|^2|X_s^{1,N}|^{2(m-1)}]\mathrm{d}s < \infty$, since $X^{1,N}$ is an interacting system of SDEs with constant diffusion and one-sided Lipschitz drift, hence has all moments. By Young's inequality,

$$|X_s^{1,N}|^{m-1}\big(\frac{1}{N}\sum_{j=1}^N |X_s^{j,N}|^2\big)^{1/2} \le \big(1-\frac{1}{m}\big)|X_s^{1,N}|^m + \frac{1}{m}\big(\frac{1}{N}\sum_{j=1}^N |X_s^{j,N}|^2\big)^{m/2}$$
$$\le |X_s^{1,N}|^m + \frac{1}{m}\frac{1}{N}\sum_{j=1}^N |X_s^{j,N}|^m,$$

similarly one obtains $|X_s^{1,N}|^{m-k} \le 1 + |X_s^{1,N}|^m$, for $k<m$. Hence we obtain,

$$\mathbb{E}_\mathbb{P}[|X_t^{1,N}|^m] \le |x_0|^m + C\int_0^t (m+\epsilon m^2)\mathbb{E}_\mathbb{P}\big[|X_s^{1,N}|^m\big] + \frac{1}{N}\sum_{j=1}^N \mathbb{E}_\mathbb{P}\big[|X_s^{j,N}|^m\big] + m + \epsilon m^2 \mathrm{d}s$$
$$\le C(C^m + m + \epsilon m^2) + C\int_0^t (m+\epsilon m^2)\mathbb{E}_\mathbb{P}\big[|X_s^{1,N}|^m\big]\mathrm{d}s$$
$$\le C(C^m + m + \epsilon m^2)\exp\big(C(m+\epsilon m^2)\big),$$

where the final bound comes from applying Grönwall's inequality. Moreover, the bound is uniform over the time variable. □

We next show that one can use Varadhan's lemma in this case.

***Lemma* 6.5.** Fix $N\in\mathbb{N}$, let $h \in \mathbb{H}_T^d$ and let Assumptions 4.2 and 4.1 hold.
Then the integrability condition in Varadhan's lemma holds for (4.9). Namely for some $\gamma > 1$

$$\limsup_{\epsilon\to 0} \epsilon \log\Big(\,\mathbb{E}_\mathbb{P}\Big[\,\exp\Big(\,\frac{\gamma}{\epsilon}\Big(2\log\Big(\tilde{G}_1(\sqrt{\epsilon}W^1,\dots,\sqrt{\epsilon}W^N)\Big) - \int_0^T \langle \dot{h}_t, \sqrt{\epsilon}\mathrm{d}W_t^1\rangle + \frac{1}{2}\int_0^T |\dot{h}_t|^2\mathrm{d}t\,\Big)\Big)\Big]\Big) < \infty.$$

***Proof.*** Using that $h\in\mathbb{H}_T^d$ is deterministic, $\dot{h}\in L_0^2(\mathbb{R}^d)$ and Cauchy-Schwarz we obtain,

$$\epsilon\log\left(\mathbb{E}_\mathbb{P}\left[\exp\left(\frac{\gamma}{\epsilon}\Big(2\log(\tilde{G}_1(\sqrt{\epsilon}W^1,\dots,\sqrt{\epsilon}W^N)) - \int_0^T\langle\dot{h}_t,\sqrt{\epsilon}\mathrm{d}W_t^1\rangle + \frac{1}{2}\int_0^T|\dot{h}_t|^2\mathrm{d}t\Big)\right)\right]\right)$$
$$\le \frac{\gamma}{2}\int_0^T|\dot{h}_t|^2\mathrm{d}t + \frac{\epsilon}{2}\log\left(\mathbb{E}_\mathbb{P}\left[\exp\left(\frac{4\gamma}{\epsilon}\log(\tilde{G}_1(\sqrt{\epsilon}W^1,\dots,\sqrt{\epsilon}W^N))\right)\right]\right)$$
$$+\frac{\epsilon}{2}\log\left(\mathbb{E}_\mathbb{P}\left[\exp\left(-\frac{2\gamma}{\epsilon}\int_0^T\langle\dot{h}_t,\sqrt{\epsilon}\mathrm{d}W_t^1\rangle\right)\right]\right).$$

It is then sufficient to show that the three terms are finite when we take $\limsup_{\epsilon\to 0}$. The first term is clearly finite by the conditions on $h$. Finiteness of the third term follows from [25, pg.16], namely for all $i \in \{1,\dots,N\}$ the stochastic integral has the distribution $\int_0^T \langle \dot h_t, \mathrm{d}W_t^i\rangle \sim \mathcal{N}(0, \int_0^T |\dot h_t|^2 \mathrm{d}t)$. Thus we obtain,

$$\limsup_{\epsilon\to 0} \frac{\epsilon}{2}\log\left(\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{2\gamma}{\epsilon}\int_0^T \sqrt{\epsilon}\langle \dot h_t, \mathrm{d}W_t^1\rangle\right)\right]\right) = \gamma^2 \int_0^T |\dot h_t|^2 \mathrm{d}t < \infty \,.$$

The final term to consider is the , $\tilde G_1$ term. By definition of $\tilde G_1$ and Assumption 4.2 we have,

$$\frac{4\gamma}{\epsilon}\log\left(\tilde G_1(\sqrt{\epsilon}W^1,\dots,\sqrt{\epsilon}W^N)\right) \le C^{\frac{4\gamma}{\epsilon}} + C^{\frac{4\gamma}{\epsilon}} \sup_{0\le t\le T} |X^{1,N}(\sqrt{\epsilon}W^1,\dots,\sqrt{\epsilon}W^N)|^{\frac{4p\gamma}{\epsilon}} \,.$$

Again in order to simplify our equations let us denote $X_t^{1,N} := X_t^{1,N}(\sqrt{\epsilon}W^1,\dots,\sqrt{\epsilon}W^N)$, applying Itô's lemma and using similar arguments to that in the proof of Lemma 6.4 we obtain

$$\begin{aligned} |X_t^{1,N}|^{\frac{4p\gamma}{\epsilon}} \le & |x_0|^{\frac{4p\gamma}{\epsilon}} + \int_0^t C\frac{4p\gamma}{\epsilon}|X_s^{1,N}|^{\frac{4p\gamma}{\epsilon}-2}\Big(|X_s^{j,N}|^2 + |X_s^{j,N}|\big(\frac{1}{N}\sum_{j=1}^N |X_s^{j,N}|^2\big)^{1/2} + 1\Big)\mathrm{d}s \\ & + C\int_0^t \frac{4p\gamma}{\epsilon}|X_s^{1,N}|^{\frac{4p\gamma}{\epsilon}-2}\langle X_s^{1,N}, \sqrt{\epsilon}\sigma \mathrm{d}W_s^1\rangle + C\frac{(4p\gamma)^2}{\epsilon}\int_0^t |X_s^{1,N}|^{\frac{4p\gamma}{\epsilon}-2}\mathrm{d}s \,. \end{aligned}$$

Taking supremum over time of both sides, noting $\sup_{0\le t\le T}\int_0^t |\cdot|\mathrm{d}s = \int_0^T |\cdot|\mathrm{d}s$ and then taking expectation one obtains,

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}}[\sup_{0\le t\le T}|X_t^{1,N}|^{\frac{4p\gamma}{\epsilon}}] \\ & \le |x_0|^{\frac{4p\gamma}{\epsilon}} + \int_0^T \frac{C}{\epsilon}\mathbb{E}_{\mathbb{P}}\Big[|X_s^{1,N}|^{\frac{4p\gamma}{\epsilon}-2}\Big(|X_s^{j,N}|^2 + |X_s^{j,N}|\big(\frac{1}{N}\sum_{j=1}^N |X_s^{j,N}|^2\big)^{1/2} + 1\Big)\Big]\mathrm{d}s \\ & \quad + \mathbb{E}_{\mathbb{P}}\Big[\sup_{0\le t\le T}\int_0^t \frac{C}{\epsilon}|X_s^{1,N}|^{\frac{4p\gamma}{\epsilon}-2}\langle X_s^{1,N}, \sqrt{\epsilon}\sigma W_s^1\rangle\Big] + \frac{C}{\epsilon}\int_0^T \mathbb{E}_{\mathbb{P}}\Big[|X_s^{1,N}|^{\frac{4p\gamma}{\epsilon}-2}\Big]\mathrm{d}s \,. \end{aligned}$$

Applying Buckholder-Davis-Gundy inequality along with Lemma 6.4, one can bound the stochastic integral as,

$$\mathbb{E}_{\mathbb{P}}\Big[\sup_{0\le t\le T}\int_0^t \frac{C}{\epsilon}|X_s^{1,N}|^{\frac{4p\gamma}{\epsilon}-2}\langle X_s^{1,N}, \sqrt{\epsilon}\sigma \mathrm{d}W_s^1\rangle\Big] \le \frac{C}{\epsilon}(1+\sqrt{\epsilon}C^{\frac{C}{2\epsilon}})\exp\left(\frac{C}{2\epsilon}\right).$$

As is carried out in Lemma 6.4, applications of Young's inequality and substituting this bound in one obtains,

$$\mathbb{E}_{\mathbb{P}}[\sup_{0\le t\le T}|X_t^{1,N}|^{\frac{4p\gamma}{\epsilon}}] \le |x_0|^{\frac{4p\gamma}{\epsilon}} + \frac{C}{\epsilon}(1+\sqrt{\epsilon}C^{\frac{C}{2\epsilon}})\exp\left(\frac{C}{2\epsilon}\right) + \frac{C}{\epsilon}\int_0^T \mathbb{E}_{\mathbb{P}}\Big[|X_s^{1,N}|^{\frac{4p\gamma}{\epsilon}}\Big] + 1\mathrm{d}s \,.$$

By using that $\mathbb{E}_{\mathbb{P}}\Big[|X_s^{1,N}|^{\frac{4p\gamma}{\epsilon}}\Big] \leq \mathbb{E}_{\mathbb{P}}\Big[\sup_{0\leq r\leq s}|X_r^{1,N}|^{\frac{4p\gamma}{\epsilon}}\Big]$ and applying Grönwall's inequality yields,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[\sup_{0\leq t\leq T}|X_t^{1,N}|^{\frac{4p\gamma}{\epsilon}}] &\leq \left(C^{\frac{C}{\epsilon}} + \frac{C}{\epsilon}(1+\sqrt{\epsilon}C^{\frac{C}{2\epsilon}})\exp\left(\frac{C}{2\epsilon}\right)\right)\exp\left(\frac{C}{\epsilon}\right)\\ &\leq \left(C^{\frac{C}{\epsilon}} + \frac{C}{\epsilon}(1+\sqrt{\epsilon}C^{\frac{C}{2\epsilon}})\right)\exp\left(\frac{2C}{\epsilon}\right).\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{\epsilon}{2}\log\left(\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{4\gamma}{\epsilon}\log(\tilde{G}_1(\sqrt{\epsilon}W^1,\dots,\sqrt{\epsilon}W^N))\right)\right]\right)\\ &\leq \frac{\epsilon}{2}\log\left(\left(C^{\frac{C}{\epsilon}} + \frac{C}{\epsilon}(1+\sqrt{\epsilon}C^{\frac{C}{2\epsilon}})\exp\left(\frac{C}{2\epsilon}\right)\right)\exp\left(\frac{C}{\epsilon}\right)\right)\\ &\leq \frac{\epsilon}{2}\log\left(C^{\frac{C}{\epsilon}}\frac{C}{\epsilon}\exp\left(\frac{2C}{\epsilon}\right)\right)\\ &\leq C(1+\log(C)) < \infty.\end{aligned}$$

To conclude, we have shown that all terms are finite and the result follows. □

***Proof of Theorem 4.6.*** The continuity of the SDE from Lemma 6.1 along with existence of a unique strong solution under Assumptions 4.1, ensure $\tilde{G}_1$ is a continuous function under Assumption 4.2.

By assumption, there exists a point $(u^1,\hat{u}) \in (\mathbb{H}_T^d)^2$ such that $\tilde{G}(u^1,\hat{u},\dots,\hat{u}) > 0$. Further, noting by Assumption 4.2,

$$\begin{aligned}2\log\left(\tilde{G}_1(u^1,\dots,u^N)\right) &\leq 2\log(C_1) + 2\log(C_2) + p\log\left(\sup_{0\leq t\leq T}|X_t^{1,N}(u^1,\dots,u^N)|^2\right)\\ &\leq 2\log(C_1) + 2\log(C_2) + p\log\left(\sup_{0\leq t\leq T}\sup_{1\leq i\leq N}|X_t^{i,N}(u^1,\dots,u^N)|^2\right).\end{aligned}$$

Then, by using Lemma 6.3 along with the observations,

$$\sup_{0\leq s\leq T}|\sigma u_s^i|^p \leq |\sigma|^p C^{p/2}\Big(\int_0^T |\dot{u}_s^i|^2\mathrm{d}s\Big)^{p/2} \quad\text{and}\quad \sup_{1\leq i\leq N}\int_0^T|\dot{u}_s^i|^2\mathrm{d}s \leq \int_0^T |\dot{u}_s|^2\mathrm{d}s,$$

we obtain

$$2\log\left(\tilde{G}_1(u^1,\dots,u^N)\right) \leq C\log(C) + C\log\Big(\int_0^T|\dot{u}_s|^2\mathrm{d}s\Big).$$

Following arguments in Lemma 7.1 of [25] we obtain the existence of maximisers. Similarly, as $\dot{h}$ only appears in (4.10) as, $\frac{1}{2}\int_0^T|\dot{h}_t - \dot{u}_t|^2\mathrm{d}t$, it is clear there exists a minimising $h$.

Moreover, continuity of $\tilde{G}$ w.r.t. the Brownian motion and finite variation assumption of $\dot{h}$ implies the exponential term in (4.9) is continuous (see Lemma 7.6 in [25]) and that the contraction principle applies. Thus to use Varadhan's lemma we only need to check the integrability condition, which is given in Lemma 6.5, hence relation (4.10) follows.

The remaining part to be proved is that (4.12) implies asymptotically optimal. This essentially relies on showing that (4.11) is a lower bound for the RHS of (4.1). Using the same arguments used to derive (4.1), one obtains the following expression for an asymptotically optimal estimator

$$\sup_{u\in(\mathbb{H}_T^d)^N}\left\{2\log(\tilde{G}_1(u^1,\dots,u^N))-\frac{1}{2}\int_0^T|\dot{u}_t^1|^2\mathrm{d}t-\frac{1}{2}\int_0^T|\dot{u}_t|^2\mathrm{d}t\right\}.$$

As (4.11) is a special case of the above supremum (taking $u^2=\cdots=u^N$) it is then clear (4.11) provides a lower bound.

Strict convexity along with the arguments in [25, page 18] yield the uniqueness which completes the proof. □

## 6.2. Decoupling algorithm - proofs for Theorem 4.5

We recall, that due to the independence of the original particle system from the SDE in question, we work on the product of two probability spaces, consequently (since $\mu^N$ will be a "realisation" coming from the space $\tilde{\Omega}$) our results are all $\tilde{\mathbb{P}}$-a.s.. As before, the first result we need to prove is that the SDE is a continuous map of the Brownian motions, the result follows Lemma 6.2.

***Lemma* 6.6.** Let $\bar{X}$ be defined as in (4.2), with coefficients and $\mu^N$ satisfying the assumptions of Theorem 4.5. Then $\bar{X}$ is a $\mathbb{P}\otimes\tilde{\mathbb{P}}$-a.s. continuous map of Brownian motion in the uniform norm.

***Proof.*** This proof is an adaptation of the proof of Lemma 6.1 making use of a suitable adaptation of Lemma 6.2. For this case, the empirical measure $\mu^N$ is exogenously given with the necessary good properties of integrability and time-continuity (see the assumptions of Theorem 4.5) and the proof of Lemma 6.2 can be simplified in several places. We justify a few points.

The application of Carathéodory's existence result for ODEs follows from the assumed joint continuity of $(t,x)\mapsto b(t,x,\mu_t^N)$ in combination with the growth assumptions. The $\|\cdot\|_\infty$ estimate follows as before, using the assumption that $\sup_{t\in[0,T]}W^{(2)}(\mu_t^N,\delta_0)<\infty$ $\tilde{\mathbb{P}}$-a.s. in inequality (6.2), where $\mu_t^N$ replaces $\nu^{\psi+g}$ and no further properties of $\mu^N$ are needed, additionally, since $\mu^N$ is now fixed and independent of the solution, there is no need for the averaging trick. For the final step, Step 4, the continuity is also simplified mimicking the arguments there and those immediately above. □

***Lemma* 6.7.** Take $\mu^N$ and $u \in \mathbb{H}_T^d$, let $b$ and $\sigma$ satisfy Assumption 4.1 and consider the following system,

$$\bar{X}_t(u_t) = x_0 + \int_0^t b\Big(s, \bar{X}_s(u_s), \mu_s^N\Big)\mathrm{d}s + \sigma u_t,$$

with some abuse of notation and we have used $\bar{X}_t(u_t) := X_t((u_s)_{s\leq t})$. Then the following bound holds,

$$|\bar{X}_t(u_t)|^2 \leq C\big(1 + \int_0^T W^{(2)}(\mu_t^N, \delta_0)^2\mathrm{d}t + \int_0^t |\sigma u_s|^{2q+2}\mathrm{d}s\big)e^C + C|\sigma u_t|^2, \qquad \tilde{\mathbb{P}} - a.s.$$

where the involved constants $C$ are uniform in time.

***Proof.*** This proof is a simplified version of the proof of Lemma 6.3 (and based Lemma 6.2). In fact, one just needs to rewrite (6.2) for this setting and the result follow after a few steps as in Lemma 6.3. □

Before proving Varadhan's lemma, the following bound is useful.

***Lemma* 6.8.** Let $\bar{X}$ be defined as in (4.2), with coefficients and $\mu^N$ satisfying the assumptions of Theorem 4.5.

Then the following bound holds on the $m$-th moment of $\bar{X}$ for $m \geq 2$, $\tilde{\mathbb{P}}$-a.s.

$$\mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}\big[|\bar{X}_t(\sqrt{\epsilon}W)|^m|\tilde{\mathcal{F}}\big] \leq C\Big(C^m + m + \epsilon m^2 + \sup_{0\leq s\leq t} W^{(2)}(\mu_s^N, \delta_0)^m\Big)\exp\big(C(m+\epsilon m^2)\big).$$

***Proof.*** We do not give further details on this proof as it is similar to that of Lemma 6.4. □

We now prove that the uniform integrability condition still holds, namely that we can still apply Varadhan's Lemma, in both settings.

***Lemma* 6.9.** Let $h \in \mathbb{H}_T^d$, then under the assumptions of Theorem 4.5 the integrability condition in Varadhan's lemma holds for (4.3). Namely, for some $\gamma > 1$ $\tilde{\mathbb{P}}$-a.s.

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}\left[\exp\left(\frac{\gamma}{\epsilon}\left(2\log(\overline{G}(\sqrt{\epsilon}W)) - \int_0^T \langle \dot{h}_t, \sqrt{\epsilon}\mathrm{d}W_t\rangle + \frac{1}{2}\int_0^T |\dot{h}_t|^2\mathrm{d}t\right)\right)\Big|\tilde{\mathcal{F}}\right] < \infty.$$

***Proof.*** The $h$ terms can be dealt with using the same arguments as before. The term we are interested in is the $G$ term. Using arguments as in the proof of Lemma 6.5, we only need to prove the following holds,

$$\limsup_{\epsilon\to 0} \frac{\epsilon}{2}\log\left(\mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}\left[\exp\left(\frac{4\gamma}{\epsilon}\log\left(G(\bar{X}(\sqrt{\epsilon}W))\right)\right)\Big|\tilde{\mathcal{F}}\right]\right) < \infty \quad \tilde{\mathbb{P}}\text{-a.s.}$$

Many of the arguments here are similar to those appearing in the proof of Lemma 6.5, we therefore give only a few steps. Although we shall often omit it, as we are working with random variables, all bounds should be understood $\tilde{\mathbb{P}}$-a.s. Again, by applying Itô's lemma on the decoupled system along with the properties of $b$ we obtain,

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}\big[\sup_{0\le t\le T}|\bar{X}_t|^{\frac{4p\gamma}{\epsilon}}|\tilde{\mathcal{F}}\big] \\
&\le |x_0|^{\frac{4p\gamma}{\epsilon}} + \int_0^T \mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}\big[C\frac{4p\gamma}{\epsilon}|\bar{X}_s|^{\frac{4p\gamma}{2}-2}\big(|\bar{X}_s|^2 + |\bar{X}_s|W^{(2)}(\mu_s^N,\delta_0) + |\bar{X}_s|\big)|\tilde{\mathcal{F}}\big]\mathrm{d}s \\
&+ \mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}\Big[\sup_{0\le t\le T}\int_0^t |\bar{X}_s|^{\frac{4p\gamma}{\epsilon}-2}\langle \bar{X}_s, \sqrt{\epsilon}\sigma\mathrm{d}W_s\rangle\Big|\tilde{\mathcal{F}}\Big] \\
&+ C\frac{(4p\gamma)^2}{\epsilon}\int_0^T \mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}\big[C\frac{4p\gamma}{\epsilon}|\bar{X}_s|^{\frac{4p\gamma}{2}-2}|\tilde{\mathcal{F}}\big]\mathrm{d}s.
\end{aligned}
$$

For the stochastic integral we use Burkholder-Davis-Gundy and Lemma 6.8. We also note by Young's inequality,

$$
|\bar{X}_s|^{\frac{4p\gamma}{\epsilon}-1}W^{(2)}(\mu_s^N,\delta_0) \le |\bar{X}_s|^{\frac{4p\gamma}{\epsilon}} + \frac{\epsilon}{4p\gamma}W^{(2)}(\mu_s^N,\delta_0)^{\frac{4p\gamma}{\epsilon}}.
$$

Noting $W^{(2)}(\mu_s^N,\delta_0)$ is an $\tilde{\mathcal{F}}$-measurable random variable, we can therefore absorb it into the constant $C$ (but it should now be viewed as a $\tilde{\mathcal{F}}$-measurable random variable). Following arguments as in Lemma 6.5, one obtains,

$$
\mathbb{E}_{\mathbb{P}\otimes\tilde{\mathbb{P}}}\big[\sup_{0\le t\le T}|\bar{X}_t|^{\frac{4p\gamma}{\epsilon}}|\tilde{\mathcal{F}}\big] \le (C^{\frac{C}{\epsilon}} + \frac{C}{\epsilon})\exp\left(\frac{2C}{\epsilon}\right).
$$

One can finish off the proof by appealing to arguments as in Lemma 6.5. □

We can now prove the second main theorem, the arguments follow similar lines to those we used to conclude the proof of Theorem 4.6.

***Proof of Theorem 4.5.*** The continuity of the SDE from Lemma 6.6 along with existence of a unique strong solution under Assumption 4.1, ensure $\overline{G}$ is a $\tilde{\mathbb{P}}$-a.s. continuous function under Assumption 4.2. We then obtain the existence of the maximiser by Lemma 6.7 and Lemma 7.1 of [25].

Moreover, the $\tilde{\mathbb{P}}$-a.s. continuity of $\overline{G}$ w.r.t. the Brownian motion and finite variation of $\dot{h}$ implies that to use Varadhan's lemma we only need to check the integrability condition, which is given in Lemma 6.9. This with Lemma 7.6 in [25] is enough to complete the proof by arguments on page 18 in [25]. □

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